JOURNAL OF COMPARATIVE ECONOMICS 3, 56–65 (1979)

The Optimal Linear Alternative to Price and Quantity Controls in the Multifirm Case[1]

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Received November 22, 1977; revised August, 1978

Karp, G., and Yohe, G. W.—The Optimal Linear Alternative to Price and Quantity Controls in the Multifirm Case

Linear control schedules in output have been shown superior in the one-firm case to either of the extreme controls—price or quantity; they optimally trade off the desirable characteristics of both extremes. When many firms are regulated, however, that superiority fades. Then total output affects expected benefits and can display a larger (or smaller) variance than the sum of individual firms' output variances (upon which expected costs depend) if costs are positively (negatively) correlated. Output variation must be discouraged (encouraged), therefore, and the linear schedule rotates toward the quantity (price) extreme. The better extreme might thereby become the best choice among all three alternatives. *J. Comp. Econ.*, Mar. 1979, 3(1), pp. 56–65. The University at Albany, Albany, New York, and Wesleyan University, Middletown, Connecticut.

Journal of Economic Literature Classification Number: 024.

The appearance of Martin Weitzman's original comparison of price and quantity controls under uncertainty (1974) has provoked a great deal of discussion in the literature of comparative economics. Authors have commented on both his treatment of uncertainty and his subsequent limiting of the center's choice to the extreme single-valued controls. Two recently published notes by Ireland (1977) and Laffont (1977) fall into the latter category, and have inspired our present pursuit of more flexible controls. We have observed, in particular, that Ireland's second-ideal price (designated IP') is actually the expected marginal-benefit schedule. While it is better than either a price or a quota in the single-firm case in which the

[1] Our sincere thanks are extended to an anonymous referee whose unusually perceptive comments on an earlier draft led us to a more significant conclusion.

0147-5967/79/010056-10$02.00/0

random elements of marginal costs and benefits are independently distributed, we were initially struck by its potential shortfall once that independence is violated. With this possibility in mind, we set out to construct yet another alternative control that is superior to the single-valued choices regardless of the correlation between benefits and costs. This new alternative, called the revenue schedule (*RS*), was developed for the single-firm case in which both Weitzman and Ireland worked; it corresponded precisely with Ireland's *IP'* when cost-benefit correlation was zero, and was shown to be superior otherwise. Were we to have stopped there, however, we would have avoided the multifirm cases in which any alternative must be shown superior to be a meaningful extension. Ireland had already intimated that such extension might cause difficulties for his *IP'* (p. 183), but all of the circumstances in which we could envision the imposition of controls included many firms. In particular, the Environmental Protection Administration is presently considering widespread use of a kind of sliding control in which taxes or subsidies are paid by or to a firm that comes in above or below a preset level of emissions.[2] Our *RS* would be a loose interpretation of these controls. Planned economies that might employ incentive schemes would do so to an entire industry. The list could go on.

The specific purpose of this paper, then, is to present the multifirm analysis of our revenue schedules. These schedules are simultaneously the optimal linear incentive functions for a profit-maximizing firm *and* the optimal combination of their logical extremes—straight price and quantity controls. Correlations in the cost conditions across firms will emerge as the crucial parameters, and we will be able to show that the schedules are superior to homogeneous control of all firms by either a price or a set of quotas. As the number of firms to be regulated becomes large, however, that superiority fades. It is therefore quite possible that the better single-valued alternative (price or quota) is still the best choice in most cases when the administrative difficulties that may be incumbent upon more complicated incentive schemes are considered. This is at least true where the simplest extension (linear schedules) is proposed.

1. A PRELIMINARY MODEL—TWO DISTINCT FIRMS

We begin by contemplating two profit-maximizing firms, represented here by cost functions $C^1(q_1,\theta_1)$ and $C^2(q_2,\theta_2)$, that produce the same good (q). The variables θ_i reflect random disturbances in costs to which the respective firms can react in making their output decisions; in making its control choices, however, the center is required to act before their values

[2] See the Conservation Foundation Report (1978) for a thorough presentation of the preliminary deliberations that preceded the current work at the EPA.

are known.[3] Benefits derived from q are meanwhile summarized by the function $B(q,\eta)$ in which η indexes the random elements of demand. It is assumed that all of the decision makers must act before the true value of η is known. As a result, the optimal set of quotas, denoted $(\hat{q}_1,\hat{q}_2)$, is characterized by the equality of expected marginal cost at each firm and expected marginal benefits overall; i.e.,

$$EC_1^1(\hat{q}_1,\theta_1) \equiv EB_1((\hat{q}_1 + \hat{q}_2), \eta), \tag{1a}$$

and

$$EC_1^2(\hat{q}_2,\theta_2) \equiv EB_1((\hat{q}_1 + \hat{q}_2), \eta). \tag{1b}$$

The analysis that follows will be much simpler if we work with the second-order Taylor approximations of the cost and benefit schedules around these quotas. To that end, we write

$$C^1(q_1,\theta_1) = a_1(\theta_1) + (C' + \alpha_1(\theta_1))(q_1 - \hat{q}_1) + \tfrac{1}{2}C^1(q_1 - \hat{q}_1)^2, \tag{2a}$$

$$C^2(q_2,\theta_2) = a_2(\theta_2) + (C' + \alpha_2(\theta_2))(q_2 - \hat{q}_2) + \tfrac{1}{2}C^2(q_2 - \hat{q}_2)^2, \tag{2b}$$

and

$$B(q,\eta) = b(\eta) + (B' + \beta(\eta))(q - (\hat{q}_1 + \hat{q}_2)) + \tfrac{1}{2}B(q - (\hat{q}_1 + \hat{q}_2))^2; \tag{3}$$

we have defined $EC_1^i(\hat{q}_i,\theta_i) \equiv C'$, $EB_1(\hat{q}_1 + \hat{q}_2, \eta) \equiv B'$ (so that $E\alpha_i(\theta_i) = E\beta(\eta) = 0$), $C^i \equiv EC_{11}^i(\hat{q}_i,\theta_i) \geqslant 0$ and $B \equiv EB_{11}(\hat{q}_1 + \hat{q}_2, \eta) \leqslant 0$ in recording the approximation.[4] Under these conditions, it can be shown that the best price order, $\tilde{p}$, is also characterized by the equality of expected marginal costs and benefits. We see from (1), then, that $\tilde{p} = C' = B'$ and from (2) that the ith firm maximizes profits in response to $\tilde{p}$ by producing $\tilde{q}_i(\theta_i) = \hat{q}_i - \alpha_i(\theta_i)/C^i$.

A welfare comparison of price and quantity controls can now be conducted in terms of the expected costs and benefits they generate. This is the foundation of Weitzman's comparative advantage (of price over quantity control), and we record it below for our two-firm example:

[3] The assumptions that have become standard within this line of research are, perhaps, troubling to some readers. We are presuming, among other things, that (a) the firms have prior knowledge of the θ_i, (b) the center knows the firms' reaction functions, and (c) transmission and reaction costs are zero. These have been discussed rather thoroughly in Yohe (1978), but the logic behind that work is simple. The relaxing of many assumptions (e.g., (a) and (b) here) has a symmetric effect on all controls, and these wash out of the comparative advantages; we are interested in *relative* levels of expected welfare. Relaxing others (e.g., (c)) could cause a shift in preference, but for reasons that lie outside the economic characteristics of the controls themselves. These assumptions are therefore made to focus our attention.

[4] It may strike the reader as odd that the first-order coefficients are not recorded as $(C_i' + \alpha_i(\theta_i))$ to distinguish the two firms with a bit more generality. Equations (11) would then imply, however, that $C_1' = C_2' = B' = C'$, and that distinction would disappear. Our approximations incorporate this observation in their formulation.

$$\Delta(\tilde{p}/\hat{q}_i) \equiv E\{B(\tilde{q}_1 + \tilde{q}_2, \eta) - C^1(\tilde{q}_1,\theta_1) - C^2(\tilde{q}_2,\theta_2)\} - E\{B(\hat{q}_1 + \hat{q}_2, \eta) - C^1(\hat{q}_1,\theta_1) - C^2(\hat{q}_2,\theta_2)\}$$

$$= \sum_{i=1}^{2}\left[\frac{(C^i + B)\sigma_i^2}{2(C^i)^2} + \frac{B\gamma}{(C^i)^2}\right] - \sum_{i=1}^{2}\frac{\omega_i}{C^i} \quad . \tag{4}$$

In transcribing (4), we have notationally defined $\omega_i \equiv \text{Cov}\,(\alpha_i(\theta_i); \beta(\eta))$, $\gamma \equiv \text{Cov}\,(\alpha_1(\theta_1); \alpha_2(\theta_2))$, and $\sigma_i^2 \equiv \text{Var}\,(\alpha_i(\theta_i))$. Since the ω_i reflect correlations between the random elements of costs and benefits, it is perhaps reasonable to presume that $\omega_i = 0$. For the sake of expositional simplicity, we will make that assumption throughout the text. It is hoped, however, that a few footnotes will convince the reader that nonzero ω_i have no effect on the qualitative results we report.[5] The second covariance reflects correlations across firms and will, by way of contrast, play a crucial role in the construction and evaluation of our revenue schedules. For the moment, though, that role is dwarfed by the impacts of the σ_i^2; these variances serve as proxies for the output variances created by $\tilde{p}$ upon which the sign of $\Delta(\tilde{p}/\hat{q})$ turns. To see this, we express (5) entirely in terms of output:[6]

$$\Delta(\tilde{p}/\hat{q}_i) = \tfrac{1}{2}B \text{ Var}\,(\tilde{q}_1 + \tilde{q}_2) + \tfrac{1}{2}C^1 \text{ Var}\,(\tilde{q}_1) + \tfrac{1}{2}C^2 \text{ Var}\,(\tilde{q}_2). \tag{5}$$

This second formulation reveals a fundamental dichotomy between the cost and benefit sides of the welfare comparison. The cost side is the sum of individual effects felt at each firm. Output variation allowed by $\tilde{p}$ is, however, cost-efficient at the firm level, and expected costs actually decrease with exaggerated variances. The benefit side, on the other hand, depends on the *total* output of both firms; as the variance in total output (and/or the curvature of the benefit schedule) increases, the expected benefits generated by $\tilde{p}$ fall. The trade-off is thus clear, and the reader is referred to Weitzman or Yohe for a more thorough discussion. We emphasize, for our present purposes, only that the potential for diversification does exist on the benefit side; output variation in one firm can actually dampen (or accentuate) the effect on total output of variation at the other whenever $\gamma < 0$ ($\gamma > 0$). It will be argued that our revenue schedules are constructed to optimally exploit this potential for diversification.

Turning now to the construction of these schedules, suppose we were to inform the ith firm that the sale of an amount q_i will generate revenue in accordance with

$$R_i(q_i) = \tilde{p}q_i + \tfrac{1}{2}x_i(q_i - \hat{q}_i)^2. \tag{6}$$

Our revenue schedule (henceforth RS) will be the corresponding marginal-revenue curve,

[5] See footnote 8, or Karp and Yohe (1977) for a more thorough treatment.

[6] For a complete derivation see Yohe (1977a, pp. 100–101).

$$RS_i(q_i) = \bar{p} + x_i(q_i - \hat{q}_i), \tag{7}$$

and will elicit a profit-maximizing response of the form

$$q_i^{R}(\theta_i) = \hat{q}_i - \frac{\alpha_i(\theta_i)}{C^i - x_i}. \tag{8}$$

If the center were to issue these schedules to both firms, the best combination of slopes, denoted $(\hat{x}_1,\hat{x}_2)$, would be the solution to simple cost-benefit problem:

$$\max_{x_1;x_2} E\{B((q_1^{R}(\theta_1) + q_2^{R}(\theta_2)), \eta) - C^1(q_1^{R}(\theta_1),\theta_1) - C^2(q_2^{R}(\theta_2),\theta_2)\}. \tag{9}$$

The appropriate first-order conditions for a maximum in (9) ultimately require that[7]

$$(C^1 - \hat{x}_1) = \frac{(C^1 - B)(C^2 - B)\sigma_2^2\sigma_1^2 - B^2\gamma}{\sigma_1^2\sigma_2^2(C^2 - B) + B\gamma\sigma_2^2} \tag{10a}$$

and

$$(C^2 - \hat{x}_2) = \frac{(C^2 - B)(C^1 - B)\sigma_2^2\sigma_1^2 - B^2\gamma}{\sigma_1^2\sigma_2^2(C^1 - B) + B\gamma\sigma_1^2}. \tag{10b}$$

It is instructive, at this point, to observe that Eqs. (10) reduce to familiar forms whenever $\gamma = 0$. In that case, $\hat{x}_1 = \hat{x}_2 = B$, and we have the logical extension of Ireland's *IP'* to the two-firm case; our revenue schedules are then simply the marginal-benefit curves associated with the individual firms. Since we have assumed, as did Ireland, that $\omega_i = 0$, we should not be surprised. This case will also serve as a benchmark when we return to (10) and discuss the impact of nonzero γ on the *RS*.[8]

In the meantime, we will compare the optimal *RS* with their logical extremes. The comparative advantage of the *RS* over the optimal quotas is computed first:

[7] Equations (10) incorporate second-order conditions to choose the root appropriate for maximization. It may have struck the reader as odd that so much of the *RS* was specified in advance rather than setting the form $a_i + x_i(q_i - b_i)$ and maximizing with respect to all of the potential parameters. It is easily shown, however, that this procedure leads to the conclusions that $\hat{a}_i = \bar{p}$ and $\hat{b}_i = \hat{q}_i$. We therefore chose to work with (6) for expositional ease without loss of generality.

[8] Notice that where $\rho = 0$, $\hat{x} = B$ again and we have Ireland's *IP'*. It is also possible to argue in this context that $\omega_i \neq 0$ changes the optimal linear schedule. In such cases, it can be shown that $\hat{x} = (B\sigma^2 - C\omega + (n - 1)B\rho\sigma^2)/(\sigma^2 - \omega)$ is the optimal slope which differs from B even where $\rho = 0$. The rationale behind the difference lies in matching the output response to the *RS* with random shifts in marginal benefits. If $\omega > 0$ ($\omega < 0$), then output tends to be low (high) just when marginal benefits are high, and such variation should be discouraged (encouraged) by a more vertical (horizontal) *RS*. Indeed, the partial of $\hat{x}$ with respect to ω is negative signifying that an increase in ω makes $\hat{x}$ more negative (e.g.).

$$\Delta(RS/\hat{q}_i) = \frac{\sigma_1^2\sigma_2^2((C^2 - B)\sigma_1^2 + (C^1 - B)\sigma_2^2 + 2B\gamma)}{2((C^1 - B)(C^2 - B)\sigma_1^2\sigma_2^2 - B^2\gamma^2)} \tag{11}$$

$$= \frac{C^1}{2}\,\mathrm{Var}\,(q_1^R(\theta_1)) + \frac{C^2}{2}\,\mathrm{Var}\,(q_2^R(\theta_2)) - \frac{B}{2}\,\mathrm{Var}\,(q_1^R(\theta_1) + q_2^R(\theta_2)) \geq 0. \tag{11'}$$

While (11) is not particularly illuminating, manipulating the algebra to achieve (11′) pays great dividends. The RS is seen to be strictly preferred in every case but those in which the $q_i^R(\theta_i)$ are both constant over all states of nature (i.e., either when B is arbitrarily negative or when both C^i are arbitrarily large). Under those circumstances, the RS evoke the same output as $(\hat{q}_1,\hat{q}_2)$ in all states of nature from both firms, and indifference is reasonable. The RS otherwise successfully increase welfare over the fixed-output controls by allowing a pattern of output variation that decreases expected costs faster than it decreases expected benefits. The comparative advantage of the RS over $\tilde{p}$ is equally straightforward to analyze:

$$\Delta(RS/\tilde{p}) = \Delta(RS/\hat{q}_i) - \Delta(\tilde{p}/\hat{q}_i)$$

$$= (C^1/2)\,\mathrm{Var}\,(q_1^R(\theta_1) - \tilde{q}_i(\theta_1)) + (C^2/2)\,\mathrm{Var}\,(q_2^R(\theta_2) - \tilde{q}_2(\theta_2)) - (B/2)\,\mathrm{Var}\,(q_1^R(\theta_1) + q_2^R(\theta_2) - \tilde{q}_1(\theta_1) - \tilde{q}_2(\theta_2)) \geq 0. \tag{12}$$

This comparative advantage is also zero only when both controls produce the same output in all states of nature (i.e., (12) equals zero only when the C^i are near infinity so that $q_i^R(\theta_i) = \tilde{q}_i(\theta_i) = \hat{q}_i$ for all θ_i). In all but a few extreme cases, therefore, the RS are better because they allow a degree of efficient response to cost fluctuations on an individual firm level that is accurately balanced against the resulting total effect of such fluctuation on the benefit side.

We can develop a better understanding of precisely how this balance is achieved with the aid of a little geometry. Figures 1 and 2 illustrate the situation for the first firm without loss of generality. The quantity control is indicated by the vertical line at $\hat{q}$ and the price control by the horizontal line at $\tilde{p}$; these are the two extremes between which the RS dwell. Schedule RS_1, for example, is drawn to illustrate the case in which $\gamma = 0$ and the curvature parameters are all finite and nonzero. If cost correlations were positive ($\gamma > 0$), though, (10a) instructs us to draw the corresponding optimal RS_2 steeper (see Fig. 1). The economic rationale behind this observation lies in the pattern of output variation implied by $\gamma > 0$. Positive correlation creates output responses at the two firms that are, on the average, in the same direction relative to their means (where costly states of nature appear contemporaneously, e.g., output at *both*

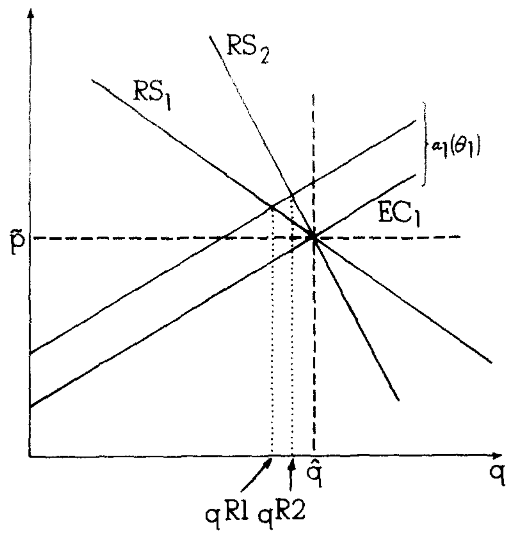

FIGURE 1

firms is lower than $\hat{q}_i$). Variation at firm 1, while cost-efficient, is therefore amplified on the benefit side by variation at firm 2 and must be diminished. Figure 1 shows that this is accomplished by a steeper RS—one that is closer to a quota. The opposite circumstance is illustrated in Fig. 2. When $\gamma < 0$, variation at firm 1 is dampened by variation at firm 2 and can be allowed to increase by issuing a flatter RS_3—one closer to a straight price control. Greater cost efficiency is thereby purchased by a small decrease in expected benefits.

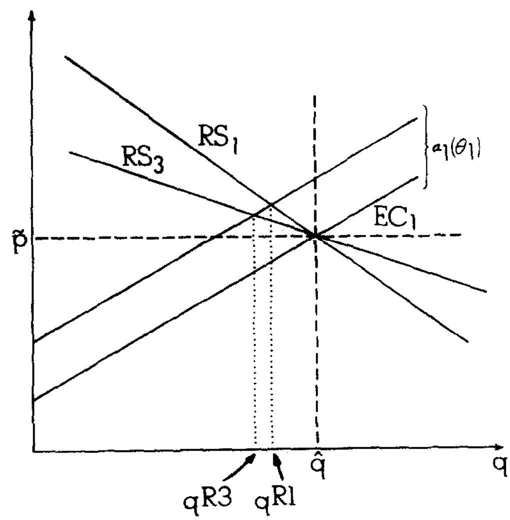

FIGURE 2

2. A MULTIFIRM EXTENSION

To extend the discussion to n completely distinct firms would burden both the analysis and the exposition with needless complication. We therefore choose to consider only a collection of identical firms. Comparative advantages of the *RS* over both price and quantity controls will be developed with an eye toward evaluating their magnitudes when the number of firms is large. We presume, then, that each firm faces the same type of cost schedule:

$$C(q_i,\theta_i) = a(\theta_i) + (C' + \alpha(\theta_i))(q_i - \hat{q}) + \tfrac{1}{2}C(q_i - \hat{q})^2,$$

where $\hat{q}$ is implicitly defined by

$$EB_1(n\hat{q},\eta) \equiv nEC_1(\hat{q},\theta_i).$$

We will also express γ in terms of its underlying correlation coefficient so that we can write

$$\text{Cov}\ (\alpha(\theta_i);\ \alpha(\theta_j)) \equiv \rho\sigma^2 \equiv \rho\ \text{Var}\ (\alpha(\theta_i))$$

for all $i \neq j$. Our notion of identical firms is thus complete. In this model, the optimal quota for every firm is $\hat{q}$, the optimal price $\tilde{p} = C$, and the optimal slope parameter for the *RS*

$$\hat{x} = B(1 + (n - 1)\rho). \tag{13}$$

The response of every firm to this *RS* is therefore simply

$$q^{R}(\theta_i) = \hat{q} - \frac{\alpha(\theta_i)}{C - \hat{x}}\ .$$

Turning to the comparative advantages, we observe that the *RS* are still superior in all but a few extreme cases. For example,

$$\Delta(RS/\hat{q}) = \frac{n}{2}\,C\ \text{Var}\ (q^{R}(\theta_i)) - \frac{1}{2}\,B\ \text{Var}\ (\sum_{i=1}^{n} q^{R}(\theta_i)) \geq 0. \tag{14}$$

The form and our interpretation of (14) are both familiar. The reader need only recall that benefits depend upon total output while costs depend upon individual firms' outputs to extrapolate our previous discussion into the context of the present model. Similarly, it is no surprise that

$$\Delta(RS/\tilde{p}) = \frac{n}{2}\,C\ \text{Var}\ (q^{R}(\theta_i) - \tilde{q}(\theta_i))$$

$$- \frac{1}{2}\,B\ \text{Var}\ (\sum_{i=1}^{n} q^{R}(\theta_i) - \sum_{i=1}^{n} \tilde{q}(\theta_i)) \geq 0. \tag{15}$$

The ability of the revenue schedule to trade efficiency for output variation is still valuable. There are, of course, circumstances in which that value is large and small, but it is always nonnegative.

All that remains is to trace the impact of changes in n, the number of firms facing regulation, on that value. We begin by emphasizing that both (14) and (15) instruct us to consider only the patterns in output variation. If we can conclude, for example, that increases in n cause the RS to swing toward either $\bar{p}$ or $\hat{q}$, we can infer that the corresponding comparative advantages decline.[9] Suppose, as a first example, that $\rho > 0$ characterizes the cost conditions across firms. Observing that

$$\lim_{n\to\infty} \hat{x}\,\Big|_{\rho>0} = -\infty,$$

we see that increasing the scope of the controls across positively correlated firms causes the optimal RS to rotate toward the vertical, the straight quota. As a result, the welfare gain achieved by imposing the optimal set of RS in lieu of quotas on a large number of firms is diminished by an increase in n.[10] The opposite conclusion emerges from slightly more complicated reasoning when $\rho < 0$. In that case, we must be mindful that the variance in total output can never become negative; i.e., we are confined in this case by the constraint that

$$\text{Var}\,\Big(\sum_{i-1}^{n} \alpha(\theta_i)\Big) = n\sigma^2 + n(n-1)\rho\sigma^2 \geqslant 0. \tag{16}$$

A derivative of (16) subsequently requires that

$$\lim_{n\to\infty} n(1 + (n-1)\rho)B = \lim_{n\to\infty} n\hat{x} \leqslant 0,$$

and since $\hat{x}$ can become nonnegative when $\rho < 0$ and n is large, it must be true that

$$\lim_{n\to\infty} \hat{x}\,\Big|_{\rho<0} = 0.$$

We therefore see that decreasing the slope of control across negatively correlated firms drives the RS downward toward the price control. The welfare gain achieved by imposing the optimal RS on a large number of firms is therefore also diminished. In fact, only when $\rho = 0$ and $\hat{x} = B$ is there no effect on the RS. When many firms are to be controlled, then it

[9] Changes in n must be accomplished without changing either $\bar{p}$ or $\hat{q}$ to insure that we are not capturing secondary changes in cost, production, or demand conditions that would distort the character of the firm. These may well be important if scope is increased by entry into the industry, but they are not where scope is increased by simply including more existing firms. In either case, this assumption allows us to focus our attention.

[10] The case in which $\rho > 0$ is precisely the case in which quotas are apt to dominate prices for large n; the choice is thus between the RS and $\hat{q}$. For the opposite reason, the choice is between the RS and $\bar{p}$ where $\rho < 0$.

can be argued that the extra effort involved in constructing a more complicated regulatory schedule may be wasted. Choosing between an optimal price standard and the best set of quotas may indeed be the best course of action.

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