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A Comparison of Price Controls and Quantity Controls Under Uncertainty

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Gary Wynn Yohe

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Abstract

A Comparison of Price Controls and Quantity Controls Under Uncertainty

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The prices-quantities comparison is conducted within a cost-benefit model in which the output of a profit maximizing enterprise is to be controlled. The regulator is constrained to the issuance of either one price order or one quantity order, determining that control specification by maximizing expected benefits minus costs. Uncertainty is introduced from three separate sources: imprecise knowledge of the cost and benefit functions themselves, the possibility that a quantity order from the center will not be filled exactly, and the chance that the quantity consumed need not equal the quantity actually produced. Any pollution example provides motivation for the final source of uncertainty. The vehicle of comparison is the comparative advantage of prices over quantities, defined to be the expected value of the algebraic difference between the level of benefits minus costs achieved with price controls and the corresponding level achieved with quantity controls.

Throughout a variety of models, including the regulation of the total output of an industry of producers of the same good, the simultaneous control of complements, substitutes, and joint products, and the regulation of an intermediate good imbedded in a vertically integrated hierarchy, the variance of output is shown to be a crucial factor. The decision-maker whose output-decision is subject to the smaller variance,

taking into account the amplifying-dampening effects of any non-zero correlations between the various sources of uncertainty, should, in general, make the quantity decision. The importance of the result to social welfare depends upon the relative curvatures of the two parts of the social loss function: benefits and costs. The propriety of mixing controls over an industry is similarly shown to hinge on the existence of a firm for which the opposite mode of control is preferred when that single firm is considered in the context of its position in the industry. That position is defined by the fraction of total output for which it is responsible and the correlation of its uncertainties with the uncertainties facing the other firms. The induced effects of output variation in one good on the marginal costs and benefits of a second nonseparable good must be considered when either of such dependent goods is to be controlled. The elasticity of substitution in production is also shown to influence the importance of the benefit side of the prices-quantities comparison when the control of an intermediate good is contemplated.

The author gratefully acknowledges the assistance of the many people who contributed to the development of this dissertation through discussion and comment on the earlier drafts. Of particular note are the members of the dissertation committee, Professors William Brainard (Chairman) and J. Michael Montias of Yale University, who generously gave of their time and their talent. Professor Martin Weitzman of the Massachusetts Institute of Technology contributed both as a motivator and as a reader, and is similarly deserving of special mention. A formal expression of gratitude is also extended to Richard Arnott for the suggestions, and the encouragement, that he offered.

Cheryl Hunt completed the typing under pressure with the patience of a saint and the fortitude of a distance runner. The spring deadline would have long since passed unmet without her help.

Finally, I wish to dedicate the dissertation to my wife, Linda, and to my parents. Their unfaltering support was the spark that kept the work progressing and makes its completion satisfying.

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Chapter One

INTRODUCTION

Most western economists prefer price control over direct quantity intervention whenever an economic activity requires outside regulation. Their preference for tariffs over any quantitative trade restriction is formally recorded in the General Agreement on Tariffs and Trade. When it was necessary to conserve fuel last winter, most economists similarly argued for tax increases instead of rationing; if rationing were to be imposed, they recommended that the ration tickets be marketable at the very least. This general preference for prices exists despite the theoretical equivalence of price and quantity controls under perfect certainty and knowledge. The presumption that price controls are better must be founded, therefore, upon the observed breakdown of one of the two qualifying assumptions. We propose to study the validity of this presumption, and while we will focus our attention primarily on the influences of uncertainty on the prices-quantities choice, we will also note the similar effects of imperfect information in several important cases.

Recent comparative systems literature has given considerable effort to the understanding of the relative merits of prices and quantities as economy-wide planning tools. When we ask whether to impose a tariff or a quota (eg.), however, we are not taking the macro perspective of this literature, but rather focusing on the regulation of a limited range of goods. Professor Martin Weitzman was the first to construct the prices-quantities comparison on this micro scale in 1973;[1] the present volume is a formal extension of his pioneering work in several illuminating directions. We begin, therefore, with a cursory review of the major arguments advanced for the general use of price controls in this more restricted context.

It is sometimes supposed that price controls have the inherent advantage that producers, being rewarded for positive profits, strive to maximize those profits. If these same producers bear any portion of costs themselves, however, they are similarly motivated to minimize costs when required to produce a given quantity. The equivalence of these two responses, assuming normal convexities, is a straightforward derivative of elementary principles.

If we expand our model to include many firms producing a single good whose total output is to be regulated, a second argument for price control proposes an "economy of information" for prices over quantities. We can easily see that the computation of an optimal price is no less involved than the corresponding computation of an optimal quantity menu. In both cases, the output of each firm must be determined to guarantee that total output is indeed equal to the prescribed level. The informational advantage, then, must lie in the need to communicate an individual quantity order to each firm rather than posting a single price order for all to see. Yet, as many authors have noted, to realistically model the control of the total output of such a collection of firms, we must postulate some kind of iterative determination of the optimal orders. The center must then receive a quantity response to the given price order

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[1]Martin Weitzman, "Prices vs. Quantities," MIT Discussion Paper, April 1973.

from each firm, just as it must receive a marginal cost response to each quantity specification. A network of informational flows between the center and each of the firms must therefore be established under either mode; it would be surprising if the additional cost of using that network twice rather than once were significant.

Were price controls universally preferable to quantity controls, one would indeed expect to see many examples of multileveled firms and governmental agencies administered by a collection of transfer prices set into a structure of personal self-interest. We observe, however, the exact opposite. This simple observation is perhaps the strongest justification for rejecting a universal presumption towards prices, and therefore the strongest reason for raising our question. In some cases, institutional constraints will surely preclude one of the two possible controls. Once we have concluded that there does not exist a transcendent rationale for harboring a general bias towards one or the other mode of control, it becomes meaningful to seek economic criteria which may be applied, on a case by case basis, to construct economically valid comparisons of prices and quantities.

To that end, we postulate a central governing body, the center, that acts in the best interest of its population by setting either price controls or quantity controls on productive activity under its jurisdiction so as to maximize benefits minus costs. In response to a price order, the periphery--the individual production units under the center's control--determines its output by maximizing profits; in response to a quantity order, it simply minimizes the costs of producing that required output. We have chosen a cost-benefit framework because it facilitates detailed interpretation of the results. It should be clear, however, that any well-behaved loss function could be employed in its place.

The first area of uncertainty that we will consider lies in the specification of the benefit and cost functions themselves. Day to day disturbances in production and other elements of pure randomness surely affect costs. We will be considering random effects that occur before the periphery makes an output decision, but after the center issues an output or price order. The periphery can therefore maximize actual profits in response to both these random effects and the price order; the center can maximize only expected benefits minus costs in determining the single order that it will issue. We require that the center be unable to change that order as the production period proceeds and the relevant random variables become known. Such a model is the logical extreme of the notion that the periphery has more information upon which to base a quantity decision. While this is surely a restrictive model, it does allow us to concentrate upon the prices-quantities comparison in the absence of the distracting complications created by any of the various control revision schemes that could be devised. Since only the center responds to benefits, on the other hand, we can treat the socially unknown aspects of the benefit function in addition to the random shocks that naturally influence benefits.

By way of illustration, consider the regulation of the sulfur dioxide emissions from a power plant. If power and sulfur dioxide are produced in fixed proportions, the simple model that we have presented to this point is perfectly applicable. Costs may be uncertain due to the uncertain reliability of technology, a dependence on weather for the efficient operation of the effluent cleansing apparatus, and the sulfur content of the fuel. Benefits are even more uncertain due to their dependence on a myriad of atmospheric variables, the ever-changing make-up of the population affected, and the imprecisely specified pollutant disage response curve. Quite obviously, the random variables influencing costs and benefits may be identical, or if different, they may be independent, positively, or negatively correlated.

It seems reasonable to suppose that the quantity ordered by the center, or by the plant manager, need not be the quantity that is actually produced. For example, shortages in inputs, equipment failures, and labor disputes only begin a list of variables that could cause output to fall short of its target, despite the best efforts of the plant administration. We will therefore attempt to incorporate these uncertainties into the model. Notice that these variables could also affect costs, so that they will appear not only as an output distortion, but also as an element in the cost function.

The third source of uncertainty with which we will deal is primarily motivated by the pollution example noted above. In that case, the amount of sulfur dioxide that anyone breathes is related to the amount emitted by the power plant by a weather-dependent diffusion equation. Thus, we have an example in which there exists a random distortion between the quantity produced, the quantity that enters the cost function, and the quantity actually consumed, the quantity that enters the benefit function. Inspired by the importance of this application, we will explore the effects of this uncertainty fully.

Finally, because they are of great practical importance, we will be concerned with errors in the observation of marginal cost. Such errors are mathematically a simple extension of the first group of uncertainties listed above, however, and do not require extensive analysis in isolation. To see this point, suppose that in simultaneous response to a price order $\tilde{p}$ and a vector of random variables $\vec{\theta}$, a firm selects its output $\tilde{q}$ by maximizing profits. In that case, $\tilde{p} = C_1(\tilde{q},\vec{\theta})$. However, if the firm is in error in measuring marginal costs, we have instead that $\tilde{p} = C_1(\tilde{q},\vec{\theta}) + \varepsilon$ where ε represents that error. Alternatively, we may assume that marginal costs are of the form $C_1(q,\vec{\theta}) + \varepsilon$. Notice that the periphery is unable to observe ε before it makes its quantity response to the price order and must, therefore, maximize the conditional expectation of profits, given the values of $\vec{\theta}$ it observes. We may further specify the problem by allowing the periphery a more accurate subjective distribution for ε than the center.

As we have already noted, we will conduct our comparison in a cost-benefit model that will be molded into a variety of forms. Following the lead of Professor Weitzman, we represent both costs and benefits by a specially devised second order Taylor approximation. We will be careful, however, to note when this assumption causes us to overlook a significant third order effect. The vehicle of comparison is the same "comparative advantage of prices over quantities" defined by Weitzman. This statistic is the expected value of the algebraic difference between the level of benefits minus costs achieved under price control and the corresponding level achieved under quantity control. If the comparative advantage is positive, then prices are preferred; if it is negative, then quantities.

After this brief introduction, Chapter 2 will present the one firm, one good case. The simplicity of this case makes it the best forum for a detailed description of the various sources of uncertainty listed above, as well as the introduction of the formal Taylor approximations of costs and benefits. The one firm assumption also affords us the opportunity of presenting geometric illustrations of the forces influencing the comparative advantage of prices. Despite the obvious narrowness of this example, it is not without significant results; in fact, the crucial influence of output variation on the prices-quantities comparison is most easily understood in the absence of cross terms and multiple producers.

Our first extension will be to increase the number of firms that produce a single output (Chapter 3). Output variation remains crucial, but we must recognize that each firm can influence only a fraction of total output. In response to this modification, one section will deal with the possibility of mixed control over the group of firms; that is, we evaluate controlling some of the firms by prices and the rest by quantities. The effect of the number of firms on the comparative advantage of prices will also be investigated.

A second extension will be the simultaneous regulation of two or more goods (Chapter 4). We will first trace the influence of independently produced complements and substitutes on the control decisions, taken jointly and individually. Secondly, we will study the regulation of joint products of the same production process, paying particular attention to the various sources of substitutability this model allows us.

Perhaps the most important generalization of our initial model is the insertion of the regulated commodity into a vertically integrated production process (Chapter 5). In this case, if the commodity to be controlled is an intermediate good, it enters the benefit function only indirectly, through the final good of the process. One should expect that the elasticity of substitution between inputs will play a significant role in determining variation in the output of the final good given any particular variation in the delivery of the controlled intermediate good. That expectation is extensively explored. We will also detect circumstances in which either mode of control creates sufficient pressure for a firm to profitably avoid that control by producing its own inputs, and provide policy alternatives to alleviate such pressure when it occurs.

A casual study of automobile emission standards will be conducted throughout the dissertation as an expanding example of the potential applicability of the analysis in each step of its completion. Despite its casual nature, however, this study will simultaneously demonstrate that automobile emission control is a tractible and timely application that should be pursued with far greater precision.

Chapter Two

THE REGULATION OF ONE FIRM PRODUCING ONE GOOD

We begin our discussion with the one firm/one product case. The various sources of uncertainty that will come under our scrutiny are most easily introduced in this simple case. The fundamental economics underlying the prices-quantities comparison are similarly most accessible when they are not camouflaged by the interdependencies of more complicated models. We will find, however, that despite the obvious naiveté of the simple model, the results derived in this chapter are extremely robust. There is no sacrifice of general validity for the expositional ease afforded us by the simplicity.

Section 2.1: Uncertainty in the Cost and Benefit Functions

2.1.1: A Geometric Presentation of the Basic Weitzman Model[2]

The one firm Weitzman model postulates costs and benefits, denoted by C and B respectively, depending explicitly on two random variables, θ and η respectively, as well as the quantity of a particular good, q, being produced:

$$B = B(q,\eta)$$

$$C = C(q,\theta)$$

[2]This subsection will reproduce the Weitzman result from a geometric basis. We do not, however, follow his convention and assume that the random disturbances of costs and benefits are independent. Nonetheless, the analysis is fundamentally Weitzman's; the geometry evolved in private discussions with Professors Montias and Brainard at Yale University.

θ and η are envisioned to represent random states of nature that disturb costs and benefits. They are assumed to be ex ante unobserved disturbances in the functions as viewed by the central regulating body that are jointly distributed by $f_{\theta\eta}(\theta,\eta)$. The center will attempt to set the production of q at the socially optimal level (i.e., the level that maximizes benefits minus costs) by issuing either a quantity order or a price order, despite the uncertainty of the economic environment. The periphery, however, is capable of observing the disturbance in the cost function before it makes its profit maximizing output decision. Under this assumption, the periphery is not interested in the benefit side of the social loss function; thus η can be thought to also reflect imprecise knowledge of the benefit function. The situation we have described is the logical extreme of the notion that the periphery possesses better information about costs than the center simply because it is closer to the actual production process. It remains only to specify the mathematical characterization of the model. We assume that

$$B_{11}(q,\eta) < 0; \quad \forall(q,\eta);$$

$$C_{11}(q,\theta) > 0; \quad \forall(q,\theta);$$

$$C_1(q,\theta) \ \& \ B_1(q,\eta) > 0; \quad \forall(q,\theta) \ \& \ (q,\eta);$$

$$B_1(0,\eta) > C_1(0,\theta); \quad \forall(\theta,\eta); \text{ and}$$

$$\exists M \text{ such that } q > M \text{ implies } B_1(q,\eta) < C_1(q,\theta)$$
$$\text{for any } (\theta,\eta).$$

The first best instruments of control are contingency messages

$p^*(\theta,\eta)$ and $q^*(\theta,\eta)$ that satisfy the condition that

$$B_1[q^*(\theta,\eta),\eta] = C_1[q^*(\theta,\eta),\theta] = p^*(\theta,\eta).$$

These contingency orders have clearly transformed ex ante uncertainty into ex post certainty. Observe further that the quantity decisions under both modes of control are characterized by price equals marginal cost.[3] The two means of regulation are therefore perfectly equivalent.

Practical considerations render such contingency messages entirely infeasible, and we turn to a "second best" problem of finding the single price or quantity order that optimally regulates production. The socially optimal quantity instrument, $\hat{q}_o$, then maximizes expected benefits minus expected costs and is characterized by

$$\begin{aligned} E(\text{Marginal Cost}) &= E[C_1(\hat{q}_o,\theta)] \\ &= E[B_1(\hat{q}_o,\eta)] \\ &= E(\text{Marginal Benefits}). \end{aligned}$$

To determine the optimal price order, the center must know how the peripheral firm will react to any given price, p. That firm still reacts precisely to θ in making its output decision $q(\theta,p)$, and maximizes profits by setting

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[3]Defining $\tilde{q}^*[p^*(\theta,\eta),\theta]$ to be the profit maximizing quantity to $p^*(\theta,\eta)$, we have that

$$C_1[\tilde{q}^*(p^*[\theta,\eta],\theta)] = p^*(\theta,\eta) = C_1[q^*(\theta,\eta),\theta].$$

The strict concavity of costs then implies that $\tilde{q}^*[p^*(\theta,\eta),\theta]$ is precisely $q^*(\theta,\eta)$.

$$C_1[q(p,\theta),\theta] = p. \qquad (2.1.1)$$

We may summarize this behavior by writing the firm's price reaction function:

$$q(p,\theta) \equiv h(p,\theta).$$

We have assumed that the center knows this function and selects the price order, $\tilde{p}$, that will again maximize expected benefits minus costs. The first order condition of this maximization can be reduced to

$$\tilde{p} = \frac{EB_1[h(\tilde{p},\theta),\eta] \cdot h_1(\tilde{p},\theta)}{Eh_1(\tilde{p},\theta)} \qquad (2.1.2)$$

by using (2.1.1).

Given the optimal price order implicitly defined by (2.1.2), the profit maximizing output is $\tilde{q}(\theta) \equiv h(\tilde{p},\theta)$. Note that except in cases of negligible probability, price control will surely yield a different level of production than will the corresponding quantity control. In addition, except in cases of zero probability,

$$B_1(\hat{q}_o,\eta) \neq C_1(\hat{q}_o,\theta), \text{ and}$$

$$B_1[\tilde{q}(\theta),\eta] \neq C_1[\tilde{q}(\theta),\theta]$$

for the duration of the planning period; that is, without extraordinarily good fortune, neither mode of control yields an <u>ex post</u> optimum output. The entire question is therefore reduced to determining which of the two comes closer. Weitzman responds to that query by computing the "comparative advantage of prices over quantities," designated by Δ in the following definition:

$$E[B(\tilde{q}[\theta],\eta)-C(\tilde{q}[\theta],\theta)] - E[B(\hat{q}_o,\eta)-C(\hat{q}_o,\theta)] \equiv \Delta \qquad (2.1.3)$$

When Δ is positive, prices are preferred; when it is negative, quantities.

Expanding both the cost and benefit functions around $\hat{q}_o$ renders the mathematics of computing Δ tractible. We assume that the variances of the random variables are sufficiently small to justify halting both approximations after three terms. Throughout the dissertation, however, attempts will be made to discuss any economically significant third order effects that are missed by our second order equations. In summary, we assume that

$$C(q,\theta) = a(\theta) + [C'+\alpha(\theta)](q-\hat{q}_o) + \frac{1}{2} C_{11}(q-\hat{q}_o)^2 \qquad (2.1.4a)$$

where we define

$$a(\theta) \equiv C(\hat{q}_o,\theta),$$

$$C' \equiv E[C(\hat{q}_o,\theta)], \text{ and}$$

$$\alpha(\theta) \equiv C_1(\hat{q}_o,\theta) - C'.$$

The form of the benefit function is similar:[4]

$$B(q,\eta) = b(\eta) + [B'+\beta(\eta)](q-\hat{q}_o) + \frac{1}{2} B_{11}(q-\hat{q}_o)^2. \qquad (2.1.4b)$$

Notice that implicit in equations (2.1.4) is the assumption that

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[4]We similarly define $b(\eta) = B(\hat{q}_o,\eta)$, $B' = EB_1(\hat{q}_o,\eta)$, and $[B_1(\hat{q}_o,\eta)-B']$ to be $\beta(\eta)$.

$C_{11}(\hat{q}_o,\theta)$ and $B_{11}(\hat{q}_o,\eta)$ are independent of the random variables θ and η, respectively. In a neighborhood of $\hat{q}_o$, marginal costs and marginal benefits are therefore assumed to be linear functions of q in which uncertainty appears only in the verticle intercepts:

$$C_1(q,\theta) = C' + \alpha(\theta) + C_{11}(q-\hat{q}_o) \qquad (2.1.5a)$$

$$B_1(q,\eta) = B' + \beta(\eta) + B_{11}(q-\hat{q}_o) \qquad (2.1.5b)$$

These assumptions will now be extended to global dimensions for the sake of geometric exposition. We should always keep in mind that we are formally constrained to a neighborhood of $\hat{q}_o$.

As a first example, suppose that benefits are devoid of uncertainty. Figure (2.1) illustrates the situation for an arbitrary value of $\bar{\theta}$. The optimal output, $q_{opt}(\bar{\theta})$, occurs at the intersection of marginal benefits and marginal costs for the given $\bar{\theta}$. Area 1 therefore represents the loss under price control by $\tilde{p}$ when $\bar{\theta}$ occurs;[5] area 2, the loss under quantity control by $\hat{q}_o$. One result is particularly striking in this example. Assume for the moment that $B_{11} = -C_{11}$. Referring to Figure (2.1), we can see that $\measuredangle(fac) = \measuredangle(fec)$ and, as a result, $\Delta(fac)$ is congruent to

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[5]The optimal price order, $\tilde{p}$, equals $EB_1(\hat{q}_o,\eta)$ for the following reason. The reaction function of the firm is given by (2.1.5a); since $h_1(p,\theta) = (1/C_{11})$, equation (2.1.2) reads

$$\tilde{p} = EB_1(\hat{q}_o + \frac{\tilde{p}-\alpha(\theta)-C'}{C_{11}},\eta)$$

so that $\tilde{p} = B' = C'$. By the definition of B', then, $EB_1(\hat{q}_o,\eta)$ equals $\tilde{p}$.

Δ(fec) by angle-angle-side. Consequently, length $\overline{ac}$ equals length $\overline{ec}$ and since $\sphericalangle$(bac) = $\sphericalangle$(dec), Δ(abc) is similarly congruent to Δ(ced). Area 1 is therefore precisely equal to area 2. Since neither B_{11} nor C_{11} vary with $\bar{\theta}$, the areas in question are equal regardless of the value taken by θ; both modes of control will always involve equal losses. We expect, therefore, that under the equality of C_{11} and $-B_{11}$, the center will be indifferent between the two methods of regulation; i.e., that $\Delta = 0$.

Figure (2.2) reverses the crucial assumption; costs are now certain and benefits are subject to a random disturbance indexed by η. The reaction function of the firm is no longer stochastic and we can see from the graph that $\hat{q}_o$ and $h(\tilde{p})$ always coincide. The variance in benefits must therefore be affecting both modes of control equally; we should not expect terms in the comparative advantage of prices to contain terms derived solely from the random variable η. In addition, we should expect that the center is indifferent between the two controls when, as in this case, costs are certain.

Turning to the general case in which uncertainty exists in both functions, we can produce the Weitzman result directly from the geometry and confirm the intuitive expectations developed in the previous two examples. Selecting $\bar{\eta}$ and $\bar{\theta}$ arbitrarily, we construct Figure (2.3) by graphing the marginal functions along side their expected values. The important points, $\hat{q}_o$, $q^{opt}(\bar{\theta},\bar{\eta})$, $\tilde{q}(\theta)$, and $\tilde{p}$, are all determined as before. Observe that length AD equals length EF because expected marginal costs and marginal costs given $\bar{\theta}$ are parallel. By definition, $C_{11} = \tan(\sphericalangle ADG)$, and we have that

FIGURE (2.1)

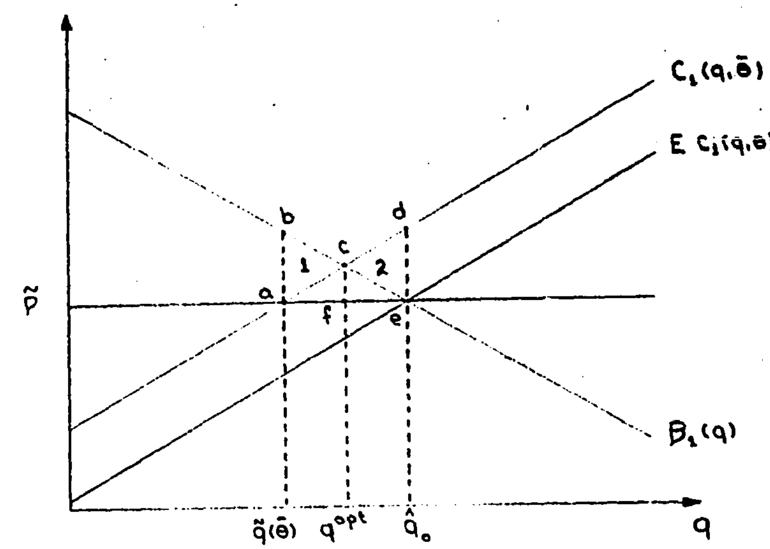
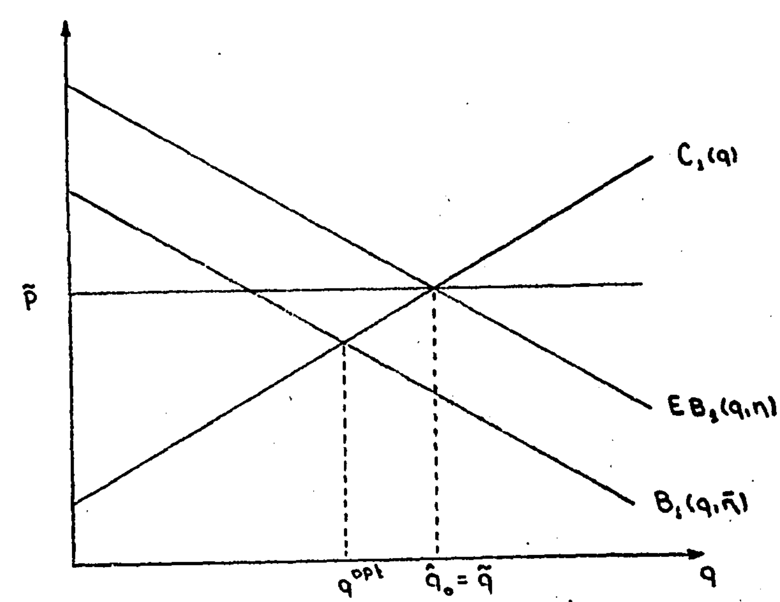


FIGURE (2.2)



$$C_{11} = \tan(\measuredangle ADG) = (AG/AD) = (AG/EF)$$

Thus,

$$C_{11} = [\alpha(\bar{\theta})/(\hat{q}_o - \tilde{q}[\bar{\theta}])].$$

Since θ was arbitrarily selected, we have demonstrated that

$$\tilde{q}(\theta) = \hat{q}_o - [\alpha(\theta)/C_{11}] \qquad (2.1.6)$$

Equation (2.1.6) is the reaction function of the firm to the random variable θ given the optimal price order $\tilde{p}$. We are now able to compute the comparative advantage of prices with the aid of Figure (2.3). On that graph, -(area 2) is the loss in benefits over costs sustained by producing at $\tilde{q}(\theta)$ rather than at $q^{opt}(\theta,\eta)$; (area 1), the loss from excess costs over benefits sustained by producing at $\hat{q}_o$. We can define the conditional comparative advantage of prices given $\bar{\theta}$ and $\bar{\eta}$ to be

$$\Delta(\bar{\theta},\bar{\eta}) \equiv (\text{-area } 2) + (\text{area } 1).$$

More precisely,

$$\Delta(\bar{\theta},\bar{\eta}) = -\int_{\tilde{q}(\bar{\theta})}^{\hat{q}_o} [B_1(q,\bar{\eta}) - C_1(q,\bar{\theta})]dq$$

The center, however, is interested in the expected value of these conditional statistics so that the complete calculation is:

$$\Delta = -E\int_{\tilde{q}(\theta)}^{\hat{q}_o} [B_1(q,\eta) - C_1(q,\theta)]\,dq$$

$$= -\left[-\frac{2C_{11}E[\alpha(\theta)]^2}{2C_{11}^2} - \left(\frac{B_{11}-C_{11}}{2C_{11}^2}\right) E[\alpha(\theta)]^2 \right] + E[\frac{-\alpha(\theta)}{C_{11}} \cdot \beta(\eta)]$$

$$= \frac{1}{2}(B_{11} + C_{11})\ \mathrm{Var}\ [\frac{-\alpha(\theta)}{C_{11}}] + \mathrm{Cov}\ [\frac{-\alpha(\theta)}{C_{11}}\ ;\ \beta(\eta)]\ . \qquad (2.1.7)$$

In both of the special examples discussed above, either costs or benefits were known with certainty. As a result, $\mathrm{Cov}[(\alpha[\theta]/C_{11});\beta(\eta)]=0$ and we observe that as predicted, Δ is indeed zero either when $B_{11} = -C_{11}$ or when costs are perfectly certain ($\mathrm{Var}[-\alpha(\theta)/C_{11}] = 0$). We can also note that random disturbances in the benefit function do not influence Δ at all. The reason for this second result lies in the reaction of the firm at the periphery to costs only. It can be demonstrated that were the firm publicly minded and set price equal to marginal benefits rather than marginal costs, $[\mathrm{Var}(-\beta[\eta]/B_{11})]$ would replace $[\mathrm{Var}(-\alpha[\theta]/C_{11})]$ in (2.1.7). In either case, however, the term in question represents the variance in output under the price regime.

We are beginning to see that the variation of output under price controls plays a crucial role in the comparison of prices and quantities. The convexity of the benefit function implies that the expected value of benefits under price regulation is less than the level of benefits that would be achieved if the mean of the price induced quantities were produced with certainty. This loss will increase as the curvature of the benefit function increases ($B_{11} \longrightarrow -\infty$).

We can illustrate these points with a graph of a simple example. Suppose that θ_1 and θ_2 are two equally probable states of nature affecting costs and that these states exhaust the realm of possibility. Assume further that $q(\theta_1) = \hat{q}_o - L$ and $q(\theta_2) = \hat{q}_o + L$. If benefits are certain, we need only compare $(1/2)[B(\hat{q}_o-L) + B(\hat{q}_o+L)]$ with $B(\hat{q}_o)$; that is, we compare point B with point A in Figure (2.4). Output variation under prices has clearly created a loss in benefits. Furthermore, as the benefit function becomes more curved, the loss is accentuated; we see easily that AB < AB'.

A similar result is true on the cost side. The concavity of the cost function implies that the expected value of costs under price regulation is greater than the level of costs that would be incurred if the mean of the price induced quantities were produced with certainty. This loss is also increased as the curvature of the cost function increases ($C_{11} \longrightarrow \infty$). When we regulate production by prices, therefore, we encounter these two losses to the degree dictated by the curvatures of the functions and the variance in output. There is, in addition, an efficiency gain in guaranteeing that price is always equal to marginal cost. These three effects are summarized in the first term of equation (2.1.7).

The second term of (2.1.7) registers the covariance of output variation under prices and the marginal benefit function. If this covariance is positive, output tends to increase as the benefit function reflects an increased desire for the good. Since this is the correct direction for output to move, we note a positive bias in favor of prices. An

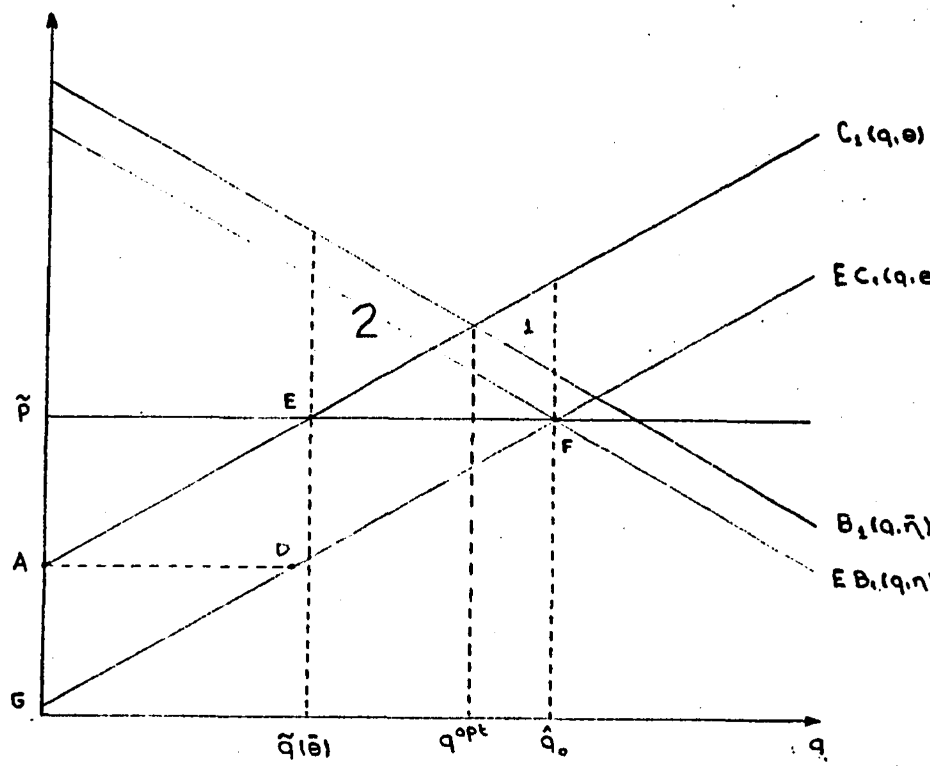
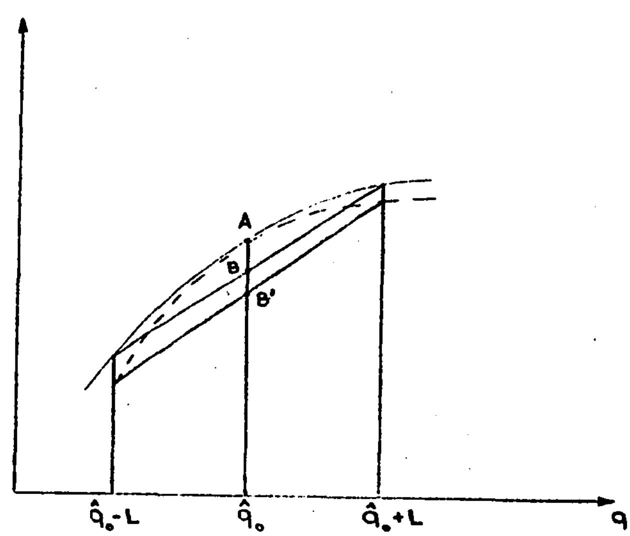


FIGURE 2.3

FIGURE 2.4



opposite bias toward quantities is similarly recognized when the covariance is negative and output tends to move against benefits.

We end this subsection by examining the implications of allowing the curvatures of the cost and benefit functions to approach their extremes. The following expressions record the results:

$$\lim_{B_{11} \to -\infty} \Delta = -\infty \qquad (2.1.8a)$$

$$\lim_{C_{11} \to 0} \Delta = -\infty \qquad (2.1.8b)$$

$$\lim_{B_{11} \to 0} \Delta = C_{11}\mathrm{Var}(\frac{-\alpha}{C_{11}}) + \mathrm{Cov}(\frac{-\alpha}{C_{11}};\beta) \qquad (2.1.8c)$$

$$\lim_{C_{11} \to \infty} \Delta = 0 \qquad (2.1.8d)$$

In light of our previous discussion on the influence of increased curvature on the loss produced by output variation, equations (2.1.8a) and (2.1.8c) are no surprise. Variation is extremely harmful when $|B_{11}|$ is large and of little consequence when $|B_{11}|$ is small. Notice that (2.1.8c) implies that the efficiency gain must always exceed the loss registered by costs due to output variation, regardless of the value of C_{11}. The role of C_{11} as the limiting factor may, however, be a small mystery; we have neglected the effect of C_{11} on the magnitude of the output variation itself.

The effect of a cost disturbance on output when marginal cost is steep (C_{11} is large) can be compared graphically to the effect of an equal disturbance when marginal cost is flat (C_{11} is small). Figure (2.5)

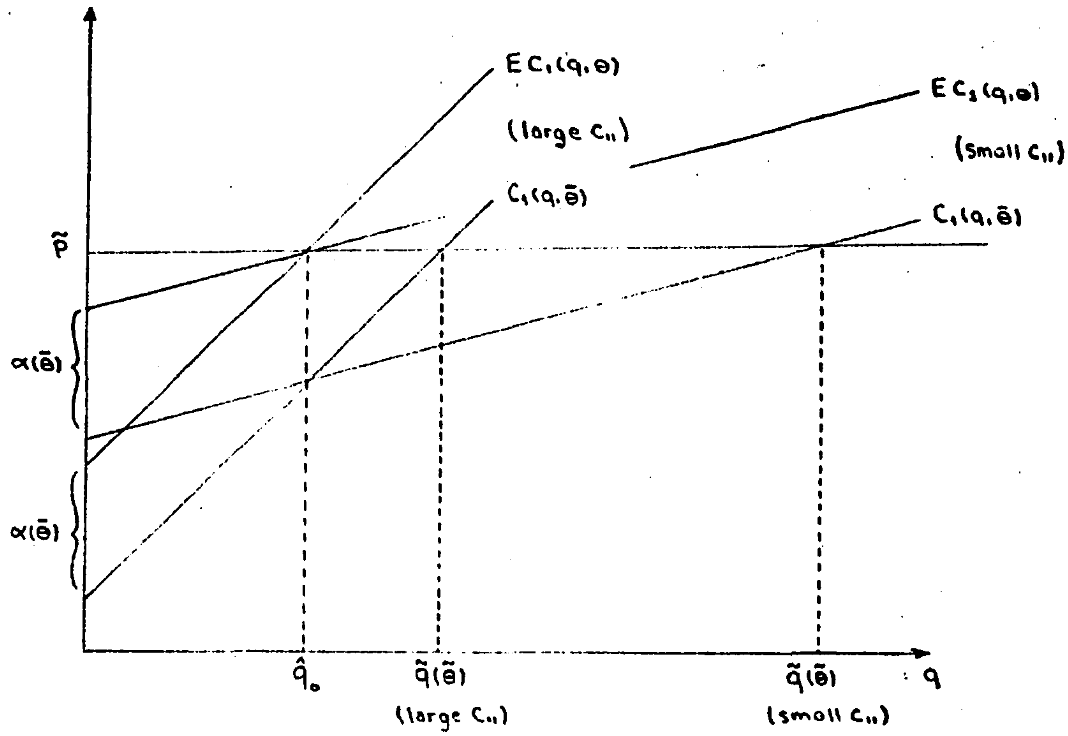


FIGURE 2.5

illustrates the situation and reveals that the smaller C_{11} becomes, the larger are the changes in output produced by cost fluctuation. The extreme cases are then quite easily explained. As marginal cost approaches the horizontal, output variation in response to cost disturbances becomes infinite and quantity control is surely preferred. The geometry has generated equation (2.1.8b). Similarly, as marginal cost nears the vertical, output variation disappears and both modes of control yield an output of $\hat{q}_o$;[6] equation (2.1.8d) accurately predicts indifference.[7]

[6] From (2.1.6) and the definition of $\alpha(\theta)$, it is clear that $E\tilde{q}(\theta)$ equals $\hat{q}_o$.

[7] As C_{11} increases arbitrarily, $\mathrm{Cov}(\frac{-\alpha}{C_{11}};\beta)$ also disappears.

In summary, there are three important cases:

1) Δ is driven negative (and thus quantity controls more preferred) as, in the neighborhood of $\hat{q}_o$, either the cost function approaches a straight line, or the benefit function becomes more sharply curved (either $C_{11} \to 0$ or $B_{11} \to -\infty$).

2) If costs and benefits are independent, Δ is driven positive (and thus price control more favored) as, in the same neighborhood, the benefit function becomes a straight line ($B_{11} \to 0$).

3) Δ is driven to zero from above (and thus the center is indifferent between controls) as the cost function becomes more sharply curved ($C_{11} \to \infty$).

Most of the specific examples that Weitzman cites conform well to the conditions of the first sentence. For a production process that is most accurately described in terms of an activity analysis model, $C_{11} = 0$ except at a finite number of points where marginal costs are undefined and statement 1 is relevant. When a pollutant is to be regulated near the critical level in the dosage response curve, B_{11} is very large in absolute value and statement 1 again applies. The simple model that we have discussed thus far can be applied to pollution examples only if the pollutant and the product of the polluting process always occur in fixed proportions so that they may be thought of as one good. The benefit function can then accurately summarize the value of the product as well as the harm of the pollutant. The very same argument vis-à-vis B_{11} can be casually applied to the case of an intermediate good with no available close substitute in production; this is the "General Motors case" and the prediction given by $\Delta < 0$ conforms well with the actual control practices of General Motors (or the Soviet economy).[8] On the other hand, if we consider a final good (an intermediate good) with a very close substitute in consumption (in production), then, in this context, B_{11} is nearly zero and statement 2 suggests that price controls are superior.

2.1.2: Errors in Observing Costs

The errors we intend to study in this subsection can be thought to arise from either incorrect measurement of, or unobserved random disturbances in, marginal costs. In either case, we represent them as additive distortions of the marginal cost schedule and index them by ϵ; that is

$$\bar{C}_1(q,\theta,\epsilon) \equiv C_1(q,\theta) + \epsilon$$

where $C_1(q,\theta)$ is the marginal cost function described in the previous subsection. Total costs are then of the form

$$\bar{C}(q,\theta,\epsilon) = C(q,\theta) + \epsilon q \qquad (2.1.9)$$

---

[8]The following explanation of the "General Motors" case has been suggested to me verbally. The heads of management at GM are worried primarily about total sales and are extremely risk averse. It is this large risk aversion that is indicated in our model by a large absolute value in B_{11}. Although the validity of this observation requires further investigation, it does illustrate an important point. The benefit function in our model need not reflect a social welfare function; rather, it can simply

for <u>both</u> the center and the peripheral firm. We assume further that $f_{\theta\eta\epsilon}(\theta,\eta,\epsilon)$ jointly distributes the three random variables. The crucial difference, however, lies neither in the source of the errors nor in their specification, but rather in the timing of their effect. The firm is unable to observe ϵ <u>before</u> it makes its output decision in response to a price order p. The firm must therefore observe θ and select its output, $q(p,\theta)$, by maximizing the conditional expectation of profits. It solves, in the context of our model, the following problem:

$$\max_{q(p,\theta)} \left\{pq(p,\theta) - \int_{\epsilon}\int_{\eta} [C(q[p,\theta],\theta) + \epsilon q(p,\theta)]\, f(\eta,\epsilon:\theta)d\eta d\epsilon\right\} \quad (2.1.10)$$

where

$$f(\eta,\epsilon:\theta) \equiv \left[\frac{f_{\theta\eta\epsilon}(\theta,\eta,\epsilon)}{\int_{\eta}\int_{e} f(\theta,\eta,e)ded\eta}\right]$$

is the conditional distribution of (η,e) given θ.

Evaluating (2.1.10) reveals that the reaction function of the firm to any price order, p, is implicitly defined by

$$p = E_{\theta}[C_1(q[p,\theta]), \theta + \epsilon]$$

$$= C' + \alpha(\theta) + C_{11}[q(p,\theta) - \hat{q}_o] + E_{\theta}e. \quad (2.1.11)$$

The notation $E_{\theta}(---)$ in (2.1.11) represents the integral operator

$$\int_{\eta}\int_{e} (---)\, f(\eta,e:\theta)ded\eta.$$

---

be the utility function of one very powerful planner, or planning agency.

Rearranging, we have that

$$q(p,\theta) = \hat{q}_o + \frac{p - c' - \alpha(\theta) - E_\theta e}{C_{11}} \equiv h(p,\theta) \qquad (2.1.12)$$

and finally that

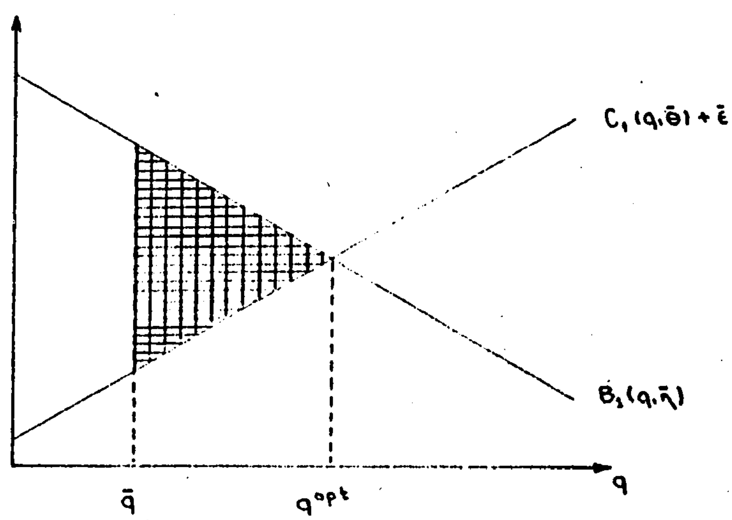
$$h_1(p,\theta) = (1/C_{11}) \qquad (2.1.13)$$

is non-stochastic.

We can now easily deduce the behavior by the center, having the complete characterization of the peripheral reaction function to any price order at hand. The loss for any quantity order, $\bar{q}$, is geometrically represented by the shaded area in Figure (2.6); $\bar{\theta}$, $\bar{\eta}$, and $\bar{\epsilon}$ have been selected arbitrarily. Algebraically, that area is

$$-\int_{\bar{q}}^{q^{opt}(\bar{\theta},\bar{\eta},\bar{\epsilon})} [B_1(q,\bar{\eta}) - C_1(q,\bar{\theta}) - \bar{\epsilon}]dq \equiv L(\bar{q};\bar{\theta},\bar{\eta},\bar{\epsilon}).$$

FIGURE 2.6



The center seeks to minimize the expected value of such losses when it sets its optimal quantity order, $\hat{q}$; the relevant first order condition is

$$0 = E\{B' + \beta_{(n)} - C' - \alpha(\theta) - e + (B_{11}-C_{11})(\hat{q} - \hat{q}_o)\}.$$

As a result,

$$\hat{q} = \hat{q}_o + \left(\frac{Ee}{B_{11}-C_{11}}\right). \qquad (2.1.14)$$

The optimal <u>price</u> order is computed by combining equations (2.1.2), (2.1.12), and (2.1.13):

$$\tilde{p} = \frac{EB_1[h(\tilde{p},\theta),\eta] \cdot h_1(\tilde{p},\theta)}{Eh_1(\tilde{p},\theta)}$$

$$= EB_1[h(\tilde{p},\theta),\eta]$$

$$= B' + B_{11}\left(\frac{\tilde{p} - c' - \alpha(\theta) \cdot E_\theta e}{C_{11}}\right).^{9} \qquad (2.1.15)$$

Since $B' = C'$, equation (2.1.15) reduces to

$$\tilde{p} = C' + \left(\frac{B_{11}}{B_{11} - C_{11}}\right) Ee ,$$

and the peripheral response curve for the optimal price order is

$$\tilde{q}(\theta) = \hat{q}_o - \left[\frac{\alpha(\theta)}{C_{11}} + \frac{E_\theta e}{C_{11}}\right] + \frac{B_{11}Ee}{C_{11}(B_{11}-C_{11})} . \qquad (2.1.16)$$

---

[9] In deriving (2.1.15), we use the fact that $E(E_\theta e) = Ee$. The proof is immediate.

The comparative advantage of prices is now available:

$$\Delta_2 = - E \int_{\tilde{q}(\theta)}^{\hat{q}} [B_1(q,\eta) - C_1(q,\theta) - \epsilon]dq$$

$$= - E \left\{ \int_{\tilde{q}(\theta)}^{\hat{q}_o - \frac{\alpha(\theta)}{C_{11}}} (B_1 - C_1 - \epsilon)dq + \int_{\hat{q}_o - \frac{\alpha}{C_{11}}}^{\hat{q}_o} (B_1 - C_1)dq - \int_{\hat{q}_o - \frac{\alpha}{C_{11}}}^{\hat{q}_o} \epsilon dq + \int_{\hat{q}_o}^{\hat{q}} (B_1 - C_1 - \epsilon)dq \right\}$$

The second integral is familiar; equation (2.1.7) records its solution:

$$\frac{1}{2}(B_{11} + C_{11}) \operatorname{Var}(\frac{-\alpha(\theta)}{C_{11}}) + \operatorname{Cov}(\frac{\cdot\alpha(\theta)}{C_{11}};\beta(\eta)).$$

We refer the reader to the last pages of subsection (2.1.1) for detailed interpretation of these terms. The other three integrals, although they are most easily solved when confronted individually, should be interpreted together. The first is the most difficult:

$$- E \int_{\tilde{q}(\theta)}^{\hat{q}_o - \frac{\alpha}{C_{11}}} (B_1 - C_1 - \epsilon)dq$$

$$= - E \left\{ [\beta_{(\eta)} - \alpha(\theta)] \frac{E_\theta \epsilon}{C_{11}} - \epsilon[\frac{E_\theta \epsilon}{C_{11}} \cdot \frac{B_{11} E\epsilon}{C_{11}(B_{11} - C_{11})}] \right.$$

$$\left. - \frac{1}{2}(B_{11} - C_{11})([\frac{E_\theta \epsilon}{C_{11}} - \frac{B_{11} E\epsilon}{C_{11}(B_{11} - C_{11})}]^2 + 2\frac{\alpha \cdot E_\theta \epsilon}{C_{11}^{\;2}}) \right\}.$$

(2.1.17)

We demonstrate in a footnote that $E(\epsilon E_\theta \epsilon)$ $E(E_\theta \epsilon)^2$.[10] Equation (2.1.17) can therefore be reduced in the following manner:[11]

---

[10]We need to show that $E(\epsilon E_\theta \epsilon) = E(E_\theta \epsilon)^2$. Notationally,

$$E_\theta \epsilon = \iint_{\eta\epsilon} \epsilon \left(\frac{f_{\theta\eta\epsilon}(\theta,\eta,\epsilon)}{\iint_{\eta\epsilon} f(\theta,\eta,\epsilon)d\epsilon d\eta}\right)d\epsilon d\eta \equiv \iint_{\eta\epsilon} \epsilon\; f(\eta,\epsilon:\theta)d\epsilon d\eta$$

Then, we can observe the following sequence of equations:

$$\begin{aligned} E(\epsilon E_\theta \epsilon) &= \iiint_{\eta\theta\epsilon} [\epsilon \iint_{\eta\epsilon} \epsilon\; f(\eta,\epsilon:\theta)d\epsilon d\eta]\; f_{\theta\eta\epsilon}(\theta,\eta,\epsilon)d\epsilon d\theta d\eta \\ &= \int_\theta [\iint_{\eta\epsilon} \epsilon\; f(\eta,\epsilon:\theta)d\epsilon d\eta] \iint_{\eta\epsilon} \epsilon\; f(\theta,\eta,\epsilon)d\epsilon d\eta d\theta \\ &= \int_\theta [(E_\theta \epsilon) \iint_{\eta\epsilon} \epsilon\; f(\eta,\epsilon:\theta)(\iint_{\eta\epsilon} f[\theta,\eta,\epsilon]d\epsilon d\eta)d\epsilon d\eta]d\theta \end{aligned}$$

since

$$f_{\theta,\eta,\epsilon}(\theta,\eta,\epsilon) \equiv f(\eta,\epsilon:\theta) \iint_{\eta\epsilon} f(\theta,\eta,\epsilon)d\epsilon d\eta.$$

Because

$$\iint_{\eta\epsilon} f(\theta,\eta,\epsilon)d\epsilon d\eta$$

is not a function of ϵ and η,

$$\begin{aligned} E(\epsilon E_\theta \epsilon) &= \int_\theta (E_\theta \epsilon)(E_\theta \epsilon) \iint_{\eta\epsilon} f_{\theta,\eta,\epsilon}(\theta,\eta,\epsilon)d\epsilon d\eta d\theta \\ &= \iiint_{\theta\eta\epsilon} (E_\theta \epsilon)^2\; f(\theta,\eta,\epsilon)d\epsilon d\eta d\theta \\ &= E(E_\theta \epsilon)^2 \end{aligned}$$

[11]We note that $E(\frac{-\alpha}{C_{11}};\frac{-E_\theta \epsilon}{C_{11}})$ is the covariance because $E(\frac{-\alpha}{C_{11}}) = 0$ and $Cov(X;Y) = E(XY - EXEY)$.

$$- E \int_{\tilde{q}(\theta)}^{\hat{q}_o - \frac{\alpha}{C_{11}}} (B_1 - C_1 - e)dq = - \{-C_{11}Cov(\frac{-\alpha}{C_{11}} ; \frac{-E_\theta e}{C_{11}}) + Cov(\frac{-E_\theta e}{C_{11}} ; \beta(\eta))$$

$$- \frac{1}{2} (B_{11} - C_{11}) \; Cov(\frac{-\alpha}{C_{11}} ; \frac{-E_\theta e}{C_{11}})$$

$$- \frac{1}{2} (B_{11} - C_{11}) \; E(\frac{E_\theta e}{C_{11}})^2 - \frac{E(E_\theta e)^2}{C_{11}}$$

$$+ \left(\frac{B_{11}}{C_{11}(B_{11}-C_{11})} - \frac{1}{2}(B_{11}-C_{11})[\frac{B_{11}{}^2}{C_{11}{}^2(B_{11}-C_{11})^2}]\right)(Ee)^2\}$$

$$= - \{-B_{11} \; Cov(\frac{-\alpha}{C_{11}} ; \frac{-E_\theta e}{C_{11}}) - Cov(-\frac{-E_\theta e}{C_{11}} ; \beta(\eta))$$

$$- \frac{1}{2}(B_{11}+C_{11}) \; E(\frac{E_\theta e}{C_{11}})^2 - \frac{1}{2} B_{11}{}^2 \; (\frac{(Ee)^2}{C_{11}(B_{11}-C_{11})})\}. \qquad (2.1.18)$$

The third integral is immediate;

$$- E \int_{\hat{q}_o - \alpha/C_{11}}^{\hat{q}_o} - e \; dq = - (Cov(e ; \frac{-\alpha}{C_{11}})$$

$$= C_{11} \; Cov(\frac{e}{C_{11}} ; \frac{-\alpha}{C_{11}}). \qquad (2.1.19)$$

Similarly for the fourth integral,

$$-E\int_{\hat{q}_o}^{\hat{q}}(B_1-C_1-\epsilon)dq = -E\{-\epsilon(\frac{E\epsilon}{B_{11}-C_{11}}) + \frac{1}{2}(B_{11}-C_{11})(\frac{E\epsilon}{B_{11}-C_{11}})^2\}$$

$$= -\frac{1}{2}[\frac{(E\epsilon)^2}{B_{11}-C_{11}}]. \qquad (2.1.20)$$

Combining equations (2.1.7), (2.1.18), (2.1.19), and (2.1.20),[12]

$$\Delta_2 = \frac{1}{2}(B_{11}+C_{11})\ Var(\frac{-\alpha}{C_{11}}) + Cov(\frac{-\alpha}{C_{11}},\beta) + \frac{1}{2}(B_{11}+C_{11})\ E(\frac{-E_\theta e}{C_{11}})$$

$$+ (B_{11}+C_{11})\ Cov(\frac{-E_\theta\epsilon}{C_{11}};\frac{-\alpha}{C_{11}}) + Cov(\frac{-E_\theta e}{C_{11}};\beta) - \frac{1}{2}(B_{11}{}^2-C_{11}{}^2)(\frac{(E\epsilon)^2}{C_{11}{}^2(B_{11}-C_{11})})$$

$$= \frac{1}{2}(B_{11}+C_{11})[Var(\frac{-\alpha}{C_{11}}) + Var(\frac{-E_\theta e}{C_{11}}) + 2\ Cov(\frac{-\alpha}{C_{11}};\frac{-E_\theta e}{C_{11}})]$$

$$+ Cov(\frac{-\alpha}{C_{11}};\beta) + Cov(\frac{-E_\theta e}{C_{11}},\beta)$$

$$= \frac{1}{2}(B_{11}+C_{11})[Var(\frac{-\alpha-E_\theta\epsilon}{C_{11}})] + Cov(\frac{-\alpha-E_\theta e}{C_{11}};\beta). \qquad (2.1.21)$$

---

[12] $Cov(\alpha;\epsilon)$ equals $Cov(\alpha;E_\theta\epsilon)$ because

$$E(\alpha E_\theta e) = E(E_\theta\alpha\cdot\epsilon) = E(\alpha\cdot e).$$

The second equality follows because $\alpha(\theta)$ is a function of neither η nor ϵ. Notice that

$$-(\frac{1}{2})(B_{11}{}^2-C_{11}{}^2)(\frac{(E\epsilon)^2}{C_{11}{}^2(B_{11}-C_{11})}) = -(\frac{1}{2})(B_{11}+C_{11})(\frac{E\epsilon}{C_{11}})^2$$

because, from elementary algebra

$$(B_{11}{}^2-C_{11}{}^2) = (B_{11}-C_{11})(B_{11}+C_{11}).$$

Observe that $(-\alpha-E_{\theta}\epsilon)/C_{11}$ represents the cumulative variation in output caused by two sources of disturbance in the cost function: actual random effects to which the peripheral firm reacts and the changes those effects create in the conditional expectations of the errors. The form of (2.1.21) is precisely that of (2.1.7); the economic influences on the comparative advantage of prices are therefore identical to those discovered in Subsection 2.1.1.

2.1.3: The Flow of Cost Information

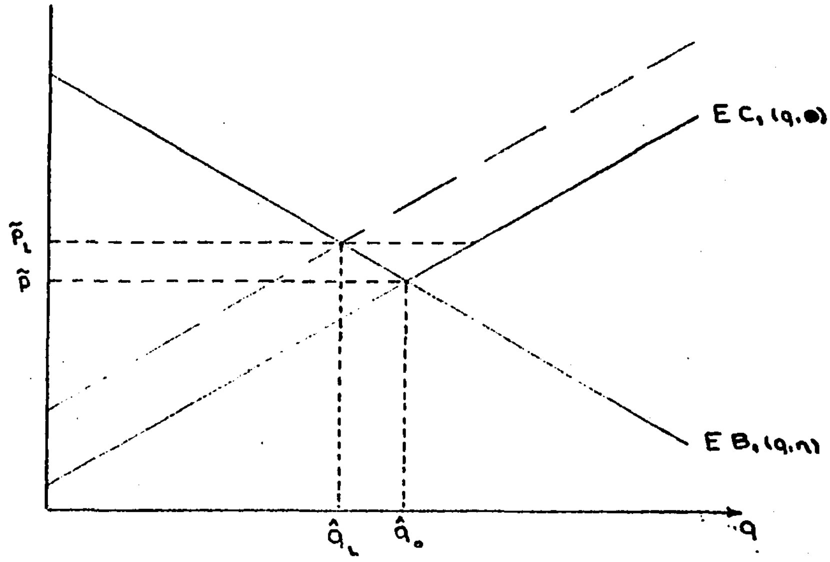
There exist two possible methods available to the center for the collection of information about costs prior to the decision making process. The relevant data could be observed by a subagency of the center, the operation of which would be a nontrivial expense. The alternative method is to depend on the peripheral firm to accurately report its costs to the center. Under the second scheme, however, there is an incentive for the firm to report higher than actual costs. Were this to happen, quantity control on the basis of that data would yield an actual output that would be socially too small; price control would yield an output that would be too large.

Figure (2.7) illustrates this point. The dotted line represents the expected marginal cost schedule that the center would use if marginal costs were systematically over-reported. The optimal quantity order, as computed then by the center, would be $\hat{q}_L < \hat{q}_o$, while the optimal price order would rise to $\tilde{p}_L > \tilde{p}$. Quantity control would therefore net $\hat{q}_L$, and price control would meanwhile average

$$E[\hat{q}_o + (\frac{\tilde{p}_L - \tilde{p} - \alpha}{C_{11}})] = \hat{q}_o + (\frac{\tilde{p}_L - \tilde{p}}{C_{11}}) > \hat{q}_o$$

because the lying firm would still place the true marginal cost equal to the higher $\tilde{p}_L$.

FIGURE 2.7



Various penalty schemes exist to abate the temptation of lying. Each has associated with it an expected dead-weight loss that must be compared on the margin with the cost of the center's observing costs for itself before a system of information collection cum penalties is imposed.

Section 2.2: Uncertainty in Output under the Quantity Regime

Consider a production activity under the quantity control of the center. There will often be circumstances that appear after the quantity order has been issued under which the required output cannot possibly be achieved, despite the best efforts of the plant manager. We therefore make a distinction between the quantity planned and ordered by the center, q_p, and the quantity actually produced, q_a. Assuming that these output distorting circumstances are random and indexing them by ξ, we presume that their effect is additive and write

$$q_a = q_p + \phi(\xi).^{13}$$

It is reasonable to assume that the random variable that distorts output must also influence costs. We therefore incorporate ξ into the argument of the cost function. The second order approximation is then[14]

$$C(q,\theta,\xi) = a(\theta,\xi) + [C' + \alpha(\theta,\xi)](q-\hat{q}_o) + \frac{1}{2} C_{11} (q-\hat{q}_o)^2 \quad (2.2.1a)$$

where

$$a(\theta,\xi) = C(\hat{q}_o,\theta,\xi),$$

$$C' = EC_1(\hat{q}_o,\theta,\xi), \text{ and}$$

$$\alpha(\theta,\xi) = C_1(\hat{q}_o,\theta,\xi) - C'.$$

The benefit function is unchanged and we assume that θ, ξ, and η are jointly distributed by $f_{\theta\xi\eta}(\theta,\xi,\eta)$.

2.2.1: The Comparative Advantage of Prices

There is an efficiency loss associated with any <u>single</u> quantity order, $\bar{q}_p$, issued by the center; following the same geometric argument used in subsection 2.1.2, we see that this loss is

---

[13]The random variable ξ can be either a scalar or a vector. To handle both cases, we represent the distortion by the function $\phi(\xi)$.

[14]To derive this expression from (2.1.4a), think of θ in subsection 2.1.1 as the vector (θ,ξ). The derivation is then immediate.
The point $\hat{q}_o$ is still defined by $EB_1(\hat{q}_o,\eta)=EC_1(\hat{q}_o,\theta,\xi)$, so we maintain the condition that $B' = C'$.

$$L(\bar{q}_p ;\cdot\bar{\theta},\bar{\xi},\bar{\eta}) = - \int_{\bar{q}_p + \phi(\bar{\xi})}^{q^{opt}(\bar{\theta},\bar{\xi},\bar{\eta})} [B_1(q,\bar{\eta})-C_1(q,\bar{\theta},\bar{\xi})]dq,$$

for arbitrarily selected values of $\bar{q}_p$, $\bar{\theta}$, $\bar{\xi}$, and $\bar{\eta}$. The center then selects the <u>optimal quantity order</u>, $\hat{q}_p$, by minimizing the expected value of these losses:

$$E[L(\hat{q}_p)] = \min_{q_p} \{E[L(q_p)]\} \;;$$

the first order condition reads:

$$0 = E\{\beta(\eta) - \alpha(\theta,\xi) + B' - C' + (B_{11}-C_{11})(\hat{q}_p+\phi(\xi) - \hat{q}_o)\}$$

Since $C' = B'$ and $E\beta(\eta) = E\alpha(\theta,\xi) = 0$, the optimal quantity order is

$$\hat{q}_p = \hat{q}_o - E\phi(\xi). \qquad (2.2.2)$$

We see that the optimal quantity order has been altered so that the expected value of actual output is still $\hat{q}_o$.

Output distortion has no effect on a firm under price control since the manager can observe ξ <u>before</u> making his output decision. The price response curve is therefore undisturbed and the <u>optimal price order</u> is still implicitly defined by

$$\tilde{p} = C'$$

Recalling that $B' = C'$, we have that $\tilde{p} = B' = C'$ and the quantity response to the optimal price remains

$$\tilde{q}(\theta,\xi) = \hat{q}_o - \frac{\alpha(\theta,\xi)}{C_{11}}. \qquad (2.2.3)$$

These computations have submerged one important ramification of the assumption that marginal costs and marginal benefits are both linear: the loss derived exclusively from the distortion in output is a constant multiple of $(B_{11}-C_{11})$. Consequently, distortions of equal magnitude but opposite direction receive the same weight in the social loss function applied by the center. To see this point, consider Figure (2.8a) in which ξ_1 and ξ_2 are defined such that $\phi(\xi_1) = \phi(\xi_2)$. When ξ_1 occurs, the loss perceived by the center is represented by [-(area 1 + area 2)]; that is,

$$L(\xi_1) = -[\tfrac{1}{2}\phi(\xi_1)\,[\phi(\xi_1)]\tan\pi + \tfrac{1}{2}[\phi(\xi_1)]^2\tan\delta]$$

$$= (B_{11}-C_{11})[\phi(\xi_1)]^2.$$

Similarly, the loss is (area 1' + area 2') when ξ_2 occurs;

$$L(\xi_2) = (B_{11}-C_{11})[\phi(\xi_2)]^2 = (B_{11}-C_{11})[\phi(\xi_1)]^2 = L(\xi_1).$$

But are large inventories as deleterious as equal shortages? If one believes not, a strong argument can be stated for the inclusion of third order terms in the approximations of the cost and benefit functions. Figure (2.8b) shows the effect of that change on the shapes of the marginal functions and the corresponding changes in the loss areas. We will explore this extension in the next subsection.

Returning to the question at hand, we are ready to compute the comparative advantage of prices:

FIGURE 2.8a

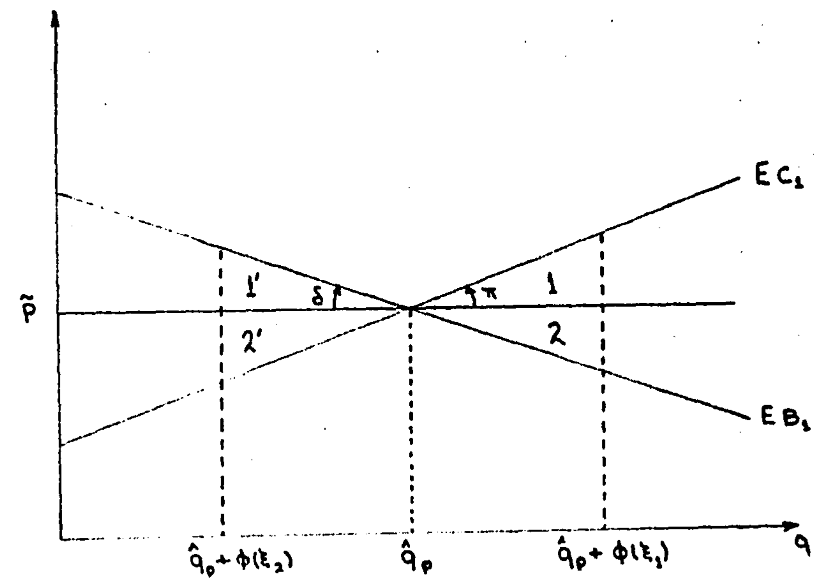
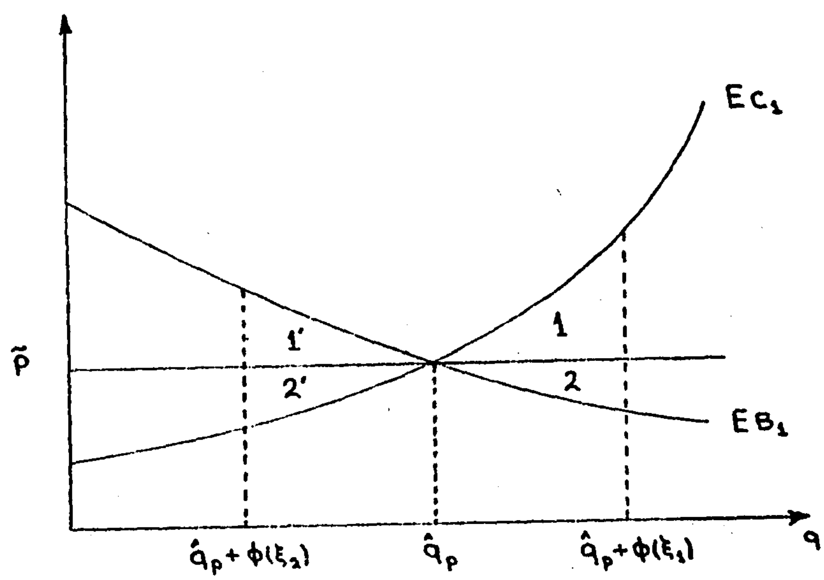


FIGURE 2.8b



$$\Delta_3 := -E \int_{\hat{q}_o - \frac{\alpha(\theta,\xi)}{C_{11}}}^{\hat{q}_o - E\phi + \phi(\xi)} [B_1(q,\eta) - C_1(q,\theta,\xi)]dq$$

$$= \frac{1}{2}(B_{11}+C_{11})\mathrm{Var}(\frac{-\alpha}{C_{11}})+\mathrm{Cov}(\frac{-\alpha}{C_{11}} ; \beta(\eta))$$

$$- \frac{1}{2}(B_{11}-C_{11})\mathrm{Var}\ \phi + \mathrm{Cov}(\alpha;\phi) - \mathrm{Cov}(\beta;\phi)$$

$$= \Delta - \frac{1}{2}(B_{11}-C_{11})\mathrm{Var}\ \phi + \mathrm{Cov}(\alpha;\phi)-\mathrm{Cov}(\beta;\phi) \qquad (2.2.4)$$

The first two terms are the comparative advantage of prices when quantity orders are filled with certainty. They will serve as a benchmark against which the remaining terms of the equation will be measured; the reader is referred to section 2 for their detailed interpretation. Output variation under quantities causes the mean of benefits to fall below (and the mean of costs to rise above) the level that would be achieved were $\hat{q}_o$ produced with certainty. Notice that unlike output variation under price control, there is no counterbalancing efficiency gain in this case. The expression $(1/2)(B_{11}-C_{11})\mathrm{Var}\phi$ records these losses and is always negative; the entire term is subtracted from the base Δ to reflect a positive bias toward prices. If $\mathrm{Cov}(\beta;\phi)$ is positive, then output under quantities tends to increase as the marginal benefits curve shifts upward. Since this is the correct direction, it constitutes a positive bias toward quantities and must also be subtracted from Δ. If $\mathrm{Cov}(\alpha;\phi)$ is positive, on the other hand, output under quantities tends to increase as costs increase; since this is the wrong direction, we should note a positive bias for prices. $\mathrm{Cov}(\alpha;\phi)$ is therefore appropriately added to the original Δ.

Table (2.1) reflects the changes these new terms create in the comparative advantage of prices when B_{11} and C_{11} assume their extreme values. One significant change occurs when C_{11} becomes arbitrarily large. Output variation is extremely harmful, in this case, and we note that prices are overwhelmingly preferred. This result is not surprising when we recall that output variation under prices disappears when C_{11} is infinite; variation under quantity control is not affected by C_{11} and is therefore shackled with huge losses.

A second change occurs when B_{11} becomes arbitrarily negative. Again we note that output variation is painfully felt, but the value of Δ_3 depends crucially on the sign of $[\mathrm{Var}(\frac{-\alpha}{C_{11}}) - \mathrm{Var}\ \phi]$. If that expression is positive, the variance of output under price control is greater than the variance of output under quantity control, and quantities are preferred. The converse, of course, is also true. Notice that the influence of C_{11} is felt even here. As C_{11} increases (i.e., as marginal costs become steeper), the variance of output under prices decreases, and it becomes more likely that prices are favored.

Table (2.1): The Pure Effect of Output Distortion Under Quantity Control

Limiting Factor	Qualifications	(section 2.2)	Δ_3
$C_{11}\to 0$	(none)	$-\infty$	$-\infty$
$C_{11}\to\infty$	(none)	0	$+\infty$
$B_{11}\to 0$	(none)	(amb)	(amb)
$B_{11}\to -\infty$	$\mathrm{Var}(\frac{-\alpha}{C_{11}})>\mathrm{Var}\ \phi$	$-\infty$	$-\infty$
	$\mathrm{Var}(\frac{-\alpha}{C_{11}})<\mathrm{Var}\ \phi$	$-\infty$	$+\infty$

2.2.2: The Impact of Third Order Terms

In using quadratic approximations of the cost and benefit functions, we have implicitly made the following assumptions:[15]

i) The underlying probability distribution is compact.

ii) The variances of the relevant random effects are sufficiently small to guarantee that the quadratic approximations yield only negligible discrepancies from the general case.

While we do not doubt the validity of the Samuelson approximation theorem, we do ask whether our making the above assumptions has obscured any significant economic results. Our interest will be focused particularly on providing an asymmetric loss function in which equal shortages and surpluses are given unequal absolute weights.[16] The desired assymmetry can be achieved by inserting third order terms into both benefits and costs:

$$C(q,\theta,\xi)=a(\theta,\xi)+[C'+\alpha(\theta,\xi)](q-\hat{q}_o)+\frac{1}{2}C_{11}(q-\hat{q}_o)^2+\frac{1}{6}C_{111}(q-\hat{q}_o)^3; \quad (2.2.5a)$$

$$B(q,\eta)=b(\eta)+[B'+\beta_{(\eta)}](q-\hat{q}_o)+\frac{1}{2}B_{11}(q-\hat{q}_o)^2+\frac{1}{6}B_{111}(q-\hat{q}_o)^3, \quad (2.2.5b)$$

where $C_{111} \equiv C_{111}(\hat{q}_o,\theta,\xi)$ and $B_{111} \equiv B_{111}(\hat{q}_o,\eta)$ are both nonstochastic and non-negative.

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[15]A compact distribution is loosely defined to be a distribution that converges to the sure outcome as some parameter goes to zero. In this case, that parameter is the variance.

[16]Recall that one of the implications of linear marginal costs and benefits is a symmetric social loss function.

By the end of this subsection, we will have shown that this complication yields little in the way of new economic insight. The analytics will have become quite involved, however, and most of the intermediate results are therefore simply stated without proof; when this is done, their intuitive foundations will be recorded in accompanying footnotes. Our first lemma demonstrates precisely how the third order terms release us from the equal valuation constraint:[17]

Lemma 1:

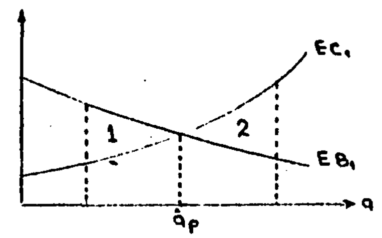
For costs and benefits represented by equations (2.2.5), if shortages are always deemed more severe than equal surpluses, then $B_{111} > C_{111}$. If shortages are less severe, then $B_{111} < C_{111}$.

For any quantity order $\bar{q}_p$, and arbitrary $\bar{\theta}$, $\bar{\xi}$, and $\bar{\eta}$, the social loss is

$$L(\bar{q}_p\ ;\ \bar{\theta},\bar{\xi},\bar{\eta}) = -\int_{\bar{q}_p+\phi(\xi)}^{q^{opt}(\bar{\theta},\bar{\xi},\bar{\eta})} [B_1(q,\bar{\eta})-C_1(q,\bar{\theta},\bar{\xi})]dq.$$

The center minimizes the expected value of this loss in selecting its optimal quantity order and therefore confronts the first order condition

[17] Suppose that ξ_1 and ξ_2 are defined so that $\phi(\xi_1) = -\phi(\xi_2)$. Then if shortages are more harmful than equal surpluses, area 1 is greater than area 2 in the graph below. Lemma 1 follows from tracing the influence of this observation on the sign of the coefficient $(B_{111}-C_{111})$.



$$0 = (B_{11}-C_{11})(\hat{q}_p + E\phi - \hat{q}_o) + \frac{1}{2}(B_{111}-C_{111})(\hat{q}_p+E\phi-\hat{q}_o)^2.$$

Therefore,

$$\hat{q}_p = \hat{q}_o - E\phi + \left[\frac{-(B_{11}-C_{11}) \pm \sqrt{(B_{11}-C_{11})^2}}{(B_{111}-C_{111})}\right]. \qquad (2.2.6)$$

The following lemma allows us to select the appropriate root:[18]

Lemma 2:

If $(B_{111}-C_{111}) > 0$, then the optimal quantity order is given by the negative root of (2.2.6). If $(B_{111}-C_{111}) < 0$, then the positive root is selected.

For the remainder of the subsection, we will assume, without loss of generality, that shortages are always worse than equal surpluses. Lemma 1 then instructs us that $(B_{111}-C_{111}) > 0$ and therefore

$$\hat{q}_p = \hat{q}_o - E(\phi),$$

since $\hat{q}_o - E(\phi) + 2\,[(B_{11}-C_{11})/(B_{111}-C_{111})]$ is the lower root.

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[18] As a function of q, expected benefits minus costs appear in two forms depending on the sign of $(B_{111}-C_{111})$.

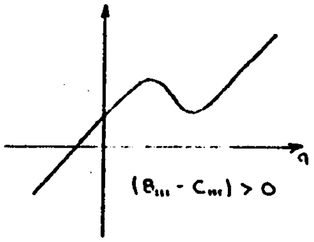

FIGURE 1

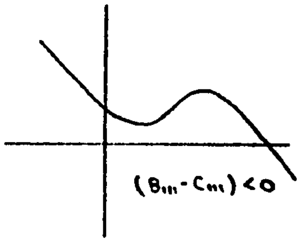

FIGURE 2

The determination of the optimal price order, $\tilde{p}$, creates even more difficulty; recall that in the general case,

$$\tilde{p} = \frac{EB_1(h,\eta) \cdot h_1}{Eh_1} \qquad (2.2.7)$$

Defining p_o to be the minimum value that marginal costs can assume at $q = 0$ over all (θ,ξ), we can combine the realistic constraint that profit maximizing output be non-negative with this definition to state equivalently that the optimal price order is

$$\tilde{p} = \max \{ p_o ; [\frac{EB_1(h,\eta) \cdot h_1}{Eh_1}]\} \qquad (2.2.8)$$

The response curve of the profit maximizing firm is then[19]

$$\tilde{q}(\theta,\xi) = \hat{q}_o + \left[\frac{-C_{11} + \sqrt{C_{11}^2 - 2C_{111}[\alpha(\theta,\xi)+C'-\tilde{p}]}}{C_{111}}\right] \qquad (2.2.9)$$

Notice that (2.2.8) implies that the radicand is always non-negative, even in the costliest state of nature. In addition, we can observe that $h_1(\tilde{p},\theta,\xi) \neq E[h_1(\tilde{p},\theta,\xi)]$; as a result, $\tilde{p} \neq E[B_1(\hat{q}_p+\phi[\xi],\eta)]$. This second

---

Constraints on the slopes of benefits and costs confine our attention to the local maxima and minima. To assure that we are maximizing E(B-C), we must select the lower root when $(B_{111}-C_{111}) > 0$ and the upper root when $(B_{111}-C_{111}) < 0$.

[19] Slope conditions imply that the firm must always choose the positive root. Indeed, [pq-C(q)] looks like Figure 2 above, and the higher root guarantees a maximum.

observation causes great difficulty in evaluating $\tilde{p}$ explicitly, so we allow $\tilde{p}$ to be implicitly defined by (2.2.8) and write $\tilde{q}(\theta,\xi) = \hat{q}_o + g[\alpha(\theta,\xi), \tilde{p}]$ for the firm's reaction function to the optimal price.

The relevant properties of $g[\alpha(\theta,\xi), \tilde{p}]$ can be deduced either from (2.2.9) or from the underlying geometry. For instance, we note immediately from (2.2.9) that $(\frac{\partial g}{\partial \alpha}) < 0$;[20] i.e., as costs rise, on the margin, output is reduced. The various moments of $g(\alpha,\tilde{p})$, on the other hand, are most easily examined in a geometric context. The first of two lemmas sets the stage:

Lemma 3:

There exists a real number, M, such that

$$g[\alpha(\theta,\xi),\tilde{p}] \leq (\frac{-\alpha(\theta,\xi)}{M}).$$

To prove this conjecture, we refer to Figure (2.9) for the case in which $\alpha(\theta,\xi) > 0$. Notice immediately that $C_{11} = \tan \pi$. Then, selecting $(\bar{\theta},\bar{\xi})$ arbitrarily, we observe

$$\begin{aligned} \text{length AB} &= \text{length EF} \\ &= -(\text{length DF}/\tan \pi) \\ &= \text{length FG} \\ &= -g[\alpha(\theta,\xi),\tilde{p}]. \end{aligned}$$

Since $(\bar{\theta},\bar{\xi})$ was arbitrary, we can define $M \equiv C_{11}$ and conclude that

$$g[\alpha(\theta,\xi),\tilde{p}] < (\frac{-\alpha(\theta,\xi)}{C_{11}}),$$

[20] We can easily see that $(\frac{\partial g}{\partial \alpha}) < 0$ since the firm always chooses the positive root.

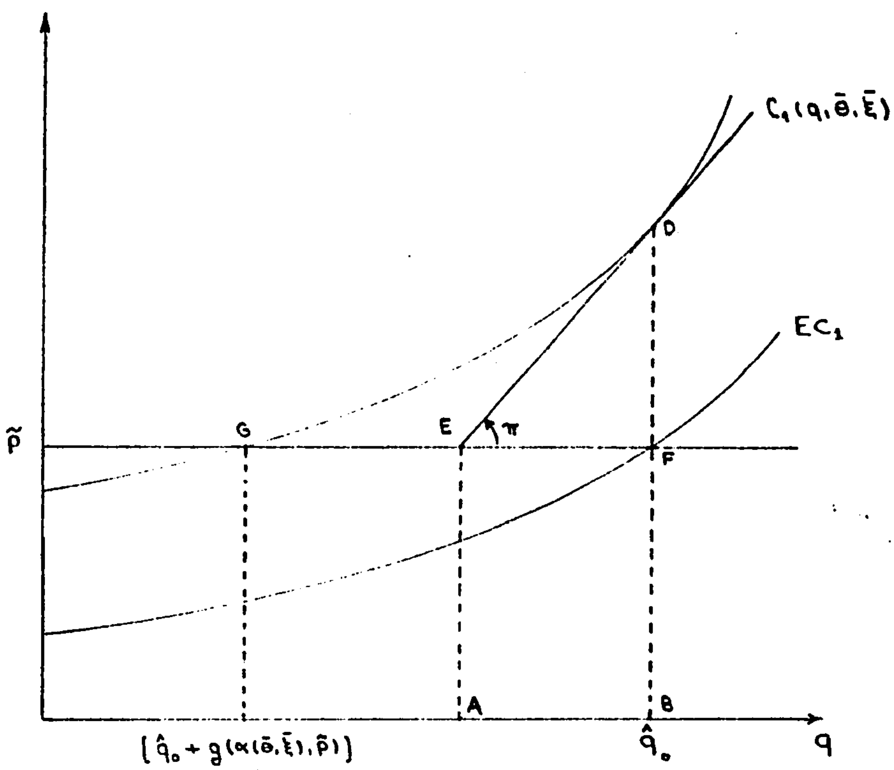
$C_1(q,\bar{\theta},\bar{\xi})$
EC_1
$\tilde{p}$
G
E
π
D
F
A
B
$[\hat{q}_0 + g(\alpha(\bar{\theta},\bar{\xi}),\tilde{p})]$
$\hat{q}_0$
q

FIGURE 2.9

for any (θ,ξ) such that $\alpha(\theta,\xi) > 0$. The case in which $\alpha(\theta,\xi) < 0$ is demonstrated in exactly the same manner.

One implication of Lemma 3 is of particular note: the output gain caused by a negative disturbance in costs is less, in absolute terms, than the output loss caused by an equal, but positive disturbance in costs. If $\alpha(\theta_1,\xi_1) = -\alpha(\theta_2,\xi_2) > 0$, then we can observe directly from the Lemma that

$$\begin{aligned} |g[\alpha(\theta_1,\xi_1),\tilde{p}]| &= - g[\alpha(\theta_1,\xi_1),\tilde{p}] \\ &> [\frac{\alpha(\theta_1,\xi_1)}{M}] \\ &= [\frac{-\alpha(\theta_2,\xi_2)}{M}] \\ &> g[\alpha(\theta_2,\xi_2),\tilde{p}] \\ &= |g[\alpha(\theta_2,\xi_2),\tilde{p}]|. \end{aligned}$$

A second lemma is more involved, and its proof is relegated to an Appendix A:

<u>Lemma 4</u>:

There exist real numbers M' and M" of the form $(a + bC_{11})$ such that

(i) for any (θ,ξ) such that $\alpha(\theta,\xi) \geq 0$,

$$g[\alpha(\theta,\xi),\tilde{p}] \geq (\frac{-\alpha(\theta,\xi)}{M'});$$

(ii) for any (θ,ξ) such that $\alpha(\theta,\xi) < 0$,

$$g[\alpha(\theta,\xi),\tilde{p}] \geq (\frac{-\alpha(\theta,\xi)}{M''}).$$

We can employ both results to put limits on the first three moments of the reaction function. For the mean, observe the following sequence of inequalities:

$$\int_{\eta} \{ \int\int_{(\theta,\xi)\epsilon X^+} (\frac{\alpha(\theta,\xi)}{M'}) f_{\theta\xi\eta}(\theta,\xi,\eta)d\xi d\theta + \int\int_{(\theta,\xi)\epsilon X^-} (\frac{\alpha(\theta,\xi)}{M''}) f_{\theta\xi\eta}(\theta,\xi,\eta)d\xi d\theta \} d\eta$$

$$= (1/M')E\alpha(\theta,\xi) + (\frac{M'-M''}{M'M''}) \iiint_{(\theta,\xi)\epsilon X^-} \alpha(\theta,\xi)f_{\theta\xi\eta}(\theta,\xi,\eta)d\xi d\theta d\eta$$

$$\geq -Eg[\alpha(\theta,\xi),\tilde{p}] \geq [\frac{E\alpha(\theta,\xi)}{M}],$$

where

$$X^+ \equiv \{(\theta,\xi): \alpha(\theta,\xi) \geq 0\} ;$$

$$X^- \equiv \{(\theta,\xi): \alpha(\theta,\xi) < 0\} .$$

Since $E\alpha(\theta,\xi) = 0$, the sequence reduces to[21]

$$(\frac{M''-M'}{M''M'}) \iiint_{X^-} \alpha(\theta,\xi)\ f(\theta,\xi,\eta)d\theta d\eta d\xi \leq g[\alpha(\theta,\xi),\tilde{p}] \leq 0 \qquad (2.2.10)$$

Recalling the general forms of M, M', and M" in terms of C_{11}, we see that all three proceed to infinity at the same rate as C_{11} becomes arbitrarily large. We conclude, therefore, that

$$\lim_{C_{11}\to\infty} E\ g[\alpha(\theta,\xi),\tilde{p}] = 0 \qquad (2.2.11)$$

[21]The left hand side is negative. This is consistent with the earlier observation that equal, but opposite, cost disturbances cause larger losses than gains in output.

An equally valid approach to this conclusion would have been to use the two lemmas directly on the reaction function to observe that $g[\alpha(\theta,\xi),\tilde{p}]$ tends to zero as C_{11} goes to infinity. Equation (2.2.11) would then follow immediately from interchanging the limit with the integral that defines the expected value operator.[22] We will illustrate this procedure in examining the variance. For any (θ,ξ) such that $\alpha(\theta,\xi) \geq 0$,

$$\left(\frac{\alpha(\theta,\xi)}{M}\right)^2 \geq g^2[\alpha(\theta,\xi),\tilde{p}] \geq \left(\frac{\alpha(\theta,\xi)}{M'}\right)^2; \qquad (2.2.12a)$$

for (θ,ξ) such that $\alpha(\theta,\xi) < 0$,

$$\left(\frac{\alpha(\theta,\xi)}{M}\right)^2 \geq g^2[\alpha(\theta,\xi),\tilde{p}] \geq \left(\frac{\alpha(\theta,\xi)}{M''}\right). \qquad (2.2.12b)$$

Therefore,

$$0 \leq \lim_{C_{11}\to\infty} [g^2(\alpha,\tilde{p})] \leq 0$$

and

$$\lim_{C_{11}\to\infty} \mathrm{Var}[g(\alpha[\theta,\xi],\tilde{p})] = \lim_{C_{11}\to\infty} (g^2(\alpha,\tilde{p}) - [Eg(\alpha,\tilde{p})]^2) = 0$$

The very same argument reveals that

$$\lim_{C_{11}\to\infty} (\mu_3[g(\alpha[\theta,\xi],\tilde{p})]) = 0$$

---

[22] In the proof given in the Appendix, we have guaranteed that the function $g[\alpha(\theta,\xi),\tilde{p}]$ is bounded from above by specifying a maximum possible price order. Since $g(\alpha,p)$ is also continuous in $\alpha(\theta,\xi)$--see equation (2.2.9)--the Lebesgue Dominated Convergence Theorem justifies our switching limits and integrals.

where $\mu_3(g[\alpha(\theta,\xi),\tilde{p}])$ is the third moment of the reaction function.

Having fully explored the relevant properties of the reaction function, we return, for the moment, to the case in which quantity orders are produced with certainty. This digression, along with the assumption that costs and benefits are independent, will allow us to discuss the influences of the third order terms in the simplest possible case. Behavior at the center is then summarized by the following equations:

$$\hat{q}_p = \hat{q}_o, \text{ and}$$

$$\tilde{p} = \frac{EB_1([\hat{q}_o+g(\alpha,\tilde{p})],\eta)\cdot g_2(\alpha,\tilde{p})}{Eg_2(\alpha,\tilde{p})}.$$

The firm's reaction curve is still $\tilde{q}(\theta,\xi) = \hat{q}_o + g[\alpha(\theta,\xi),\tilde{p}]$.[23] In this context, the comparative advantage of prices is given by:

$$\Delta_4 = -E\int_{\hat{q}_o + g(\alpha,\tilde{p})}^{\hat{q}_o} [B_1(q,\eta)-C_1(q,\theta,\xi)]dq$$

$$= -E[\alpha\cdot g(\alpha,\tilde{p})] + \frac{1}{2}(B_{11}-C_{11})\text{Var}[g(\alpha,\tilde{p})] + \frac{1}{6}(B_{111}-C_{111})[\mu_3(g[\alpha,\tilde{p}])]$$

$$+ \frac{1}{2}(B_{11}-C_{11})(Eg)^2 + \frac{1}{6}(B_{111}-C_{111})[(Eg)^3 + 3Eg \text{ Var } g]. \quad (2.2.13)$$

The extreme values of B_{11} and C_{11} provide a revealing arena in which to begin our discussion of Δ_4. Observe that these parameters still reflect the curvatures of the benefit and cost functions, respectively. We note first of all that

$$\lim_{B_{11}\to -\infty} \Delta_4 = -\infty$$

[23]In this digression, the random variable ξ remains as an index of a second source of cost disturbance.

On the other hand, we have just demonstrated that

$$\lim_{C_{11}\to\infty} E[g(\alpha,\tilde{p})] = 0,$$

$$\lim_{C_{11}\to\infty} \mathrm{Var}[g(\alpha,\tilde{p})] = 0, \text{ and}$$

$$\lim_{C_{11}\to\infty} \mu_3[g(\alpha,\tilde{p})] = 0.$$

Since $g(\alpha,\tilde{p})$ is zero at the same limit,

$$\lim_{C_{11}\to\infty} \mathrm{Cov}[\alpha;g(\alpha,\tilde{p})] = 0.$$

Equation (2.2.11) can be used to show that

$$\lim_{C_{11}\to\infty} C_{11}\mathrm{Var}[g(\alpha,\tilde{p})] = 0,$$

as well.[24] As a result, we conclude that

$$\lim_{C_{11}\to\infty} \Delta_4 = 0.$$

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[24]From equations (2.2.12),

$$C_{11}\left(\frac{\alpha(\theta,\xi)}{M'}\right)^2 \geq C_{11}g^2(\alpha,\tilde{p}) \geq C_{11}\left(\frac{\alpha(\theta,\xi)}{C_{11}}\right)^2$$

for (θ,ξ) such that $\alpha(\theta,\xi) > 0$; for (θ,ξ) such that $\alpha(\theta,\xi) < 0$,

$$C_{11}\left(\frac{\alpha(\theta,\xi)}{M''}\right)^2 \geq C_{11}g^2(\alpha,\tilde{p}) \geq C_{11}\left(\frac{\alpha(\theta,\xi)}{C_{11}}\right)^2.$$

In either case, since M' and M'' are both of the form $(a + bC_{11})$,

$$0 \geq \lim_{C_{11}\to\infty} [C_{11}g^2(\alpha,\tilde{p})] \geq 0.$$

In both cases, variation in output under prices is deemed undesirable because of large curvature in the loss function. The effect of cost disturbances on output is diminished, however, as C_{11} increases. The resulting indifference in the limiting case provides strong evidence that output variation under prices remains the powerful determinant of the comparative advantage.

Consideration of the third moment term in (2.2.13) will strengthen this conclusion even further. Suppose that positive divergences from the mean in marginal costs are larger than the corresponding negative divergences; that is, suppose that marginal costs are positively skewed. We have already noted in Lemma 3 that an increase in costs will create a larger change in output (a reduction) than will an equal decrease in costs (that will create an increase in output). Quantity shortfalls are therefore larger than surpluses. Recalling that shortfalls have been assumed more harmful than surpluses, we easily see how this bias in output variation becomes an additional liability to price regulation. The effect of cost disturbances on output is diminished, however, as C_{11} increases; in the limit, the third moment term vanishes. Observe, as a final note, that if costs were negatively skewed to a degree sufficient to create a positive output bias, then output skewness would be in the correct direction and would weigh positively on the comparative advantage of prices.

The third moment term, together with $(\frac{1}{2})(B_{11}-C_{11})\mathrm{Var}[g(\alpha,\tilde{p})]$ measures the losses that price controls incur because of output variation around a mean, $\hat{q}_o + E[g(\alpha[\theta,\xi]\tilde{p})]$. Recall that such variation results in less benefits and higher costs than certain output production at the mean.

The counterbalancing efficiency gains are captured by the covariance of α and $g(\alpha,\tilde{p})$. Since profit maximizers react correctly to changes in costs, this covariance is always negative (as costs increase, i.e. $\alpha(\theta,\xi)$ increases, output is diminished <u>ceteris paribus</u>; $g(\alpha,\tilde{p})$ falls). Only price controls allow this reaction, so the negative covariance is subtracted to provide a positive bias in the comparative of prices.

There are several remaining terms in the expression for Δ_4 that warrant at least passing mention. We can easily show that since B' = C',

$$-E\int_{\hat{q}_o + (g-Eg)}^{\hat{q}_o + (g-Eg) + g} (B_1-C_1)dq = \frac{1}{2}(B_{11}-C_{11})(Eg)^2 + \frac{1}{6}(B_{111}-C_{111})(Eg)^3 + \frac{1}{2}(B_{111}-C_{111})(Eg)\text{Var } g.$$

The last three terms in equation (2.2.13) therefore represent the extra loss attributed to price control because the expected value of output does not equal $\hat{q}_o$--the point at which expected marginal costs and benefits are equal. Notice that when $Eg < 0$, for example, that the "straight line loss," measured by $(\frac{1}{2})(B_{11}-C_{11})(Eg)^2$, is dampened by the third order term. The same curvature in the marginal functions that introduces asymmetry into social losses also tends to reduce this particular liability for prices, in this particular case. There are, to be sure, other possible cases in which the third order term amplifies the loss.

The effects that are created by the reintroduction of non-trivial output distortion under quantity control are entirely predictable. In this case, the comparative advantage of prices is given by Δ_4' below:

$$\Delta_4' = -E[\alpha \cdot g(\alpha,\tilde{p})] - E[\alpha \cdot \phi(\xi)]$$
$$+ \frac{1}{2}(B_{11}-C_{11})(\mathrm{Var}\ g(\alpha,\tilde{p}) - \mathrm{Var}\ \phi)$$
$$+ \frac{1}{6}(B_{111}-C_{111})[\mu_3(g[\alpha,\tilde{p}] - \mu_3[\phi])]$$
$$+ \frac{1}{2}(B_{11}-C_{11})(Eg)^2 + \frac{1}{6}(B_{111}-C_{111})[(Eg)^3 + 3(Eg)\mathrm{Var}\ g]. \tag{2.2.14}$$

In the limit as C_{11} becomes arbitrarily large, we now find prices overwhelmingly preferred;

$$\lim_{C_{11}\to\infty} \Delta_4' = \mathrm{Cov}(\alpha;\phi) + (\tfrac{1}{2})C_{11}\mathrm{Var}(\phi) = \infty.$$

Output variation is, of course, detrimental at this extreme, and while increasing C_{11} will decrease the variance of output under prices, it has no effect on the variance of output under quantities. Similarly,

$$\lim_{B_{11}\to -\infty} \Delta_4' = \begin{cases} +\infty; & \mathrm{Var}[g(\alpha,\tilde{p})] < \mathrm{Var}(\phi) \\ -\infty; & \mathrm{Var}[g(\alpha,\tilde{p})] > \mathrm{Var}(\phi); \end{cases}$$

the relative degree of output variation is again crucial. Output variation is likely to be less under prices than under quantities for large values of C_{11}, and price controls are therefore to be favored. Conversely, when C_{11} is small, the effects of costs disturbances in a price regime will likely dominate and quantities will be preferred.

The third moment term in (2.2.14) functions precisely as it did in (2.2.13). The dichotomy of effects--$\phi(\xi)$ influencing only quantity controls and $g(\alpha,\tilde{p})$ influencing only the price mode--generates the opposite signs. The final new term is the covariance of the random variable influencing output under quantities and the random changes in marginal costs; we have seen it before. If that covariance is negative, output under the quantity mode of control tends to decrease as marginal costs are increasing. Since this is the correct direction, we observe a positive bias toward the quantity mode. Output tends to move in the wrong direction and produces a corresponding liability to quantities if the covariance is positive.

The lesson of this subsection is, at this point, intuitively clear: the third order terms create a great deal of mathematical havoc and reveal very little new economics. The new effects that were revealed could have been demonstrated just as well by heuristic reasoning. We return, therefore, to the realm of linear marginal functions. Any significant third order effects that we thereby submerge will be deduced heuristically in the context of the geometry that originally inspired this subsection.

2.2.3: The Impact of a Capacity Constraint

We have noted in subsection 2.2.1 that central planners "totally adapt" their quantity orders to the output distortion that they face; that is, the order is specified so that the output mean is precisely equal to the output level that would be required were quantity orders filled with certainty.[25] It is often argued, however, that in the face

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[25] Equation (2.2.2) observes that $\hat{q}_p = \hat{q}_o - E\phi$. Actual output, therefore, has a mean given by $E[\hat{q}_o - E\phi + \phi(\xi)] = \hat{q}_o$.

of a capacity constraint, planners hedge, in the short run, against the severe cost penalties of producing above normal capacity by not adapting their quantity orders completely. We can conceive of these cost increases emerging from a variety of sources; overtime wages and increased maintenance on overworked and accelerated machinery are but two entries in a long list. In this brief subsection, we examine this conjecture and infer its effect on the prices-quantities comparison by applying a previous result.

There is no need to become swamped by equations in this study; geometric reasoning will suffice. We can incorporate a capacity constraint into our linear model in the following manner. Define a point q^{cap} at which marginal costs become suddenly steeper for all states of nature. Figure (2.10) illustrates our definition for several arbitrary values of (θ,ξ). If there exists a state of nature such that $\hat{q}_o + \phi(\xi)$ exceeds q^{cap} (a reasonable formalization of "facing a capacity constraint"), then that state is burdened with a penalty of higher costs. The shaded area in Figure (2.11) illustrates this penalty graphically. These penalties achieve an increase in expected costs of

$$E \left\{ \int_{\hat{q}_o + \phi(\xi)}^{q^{cap}} (C_{11}^1 + C_{11}^2)(q - \hat{q}_o)dq \right\} > 0$$

As a result, the optimal quantity order is reduced;

$$\hat{q}_p = \hat{q}_o - E(\phi) - A,$$

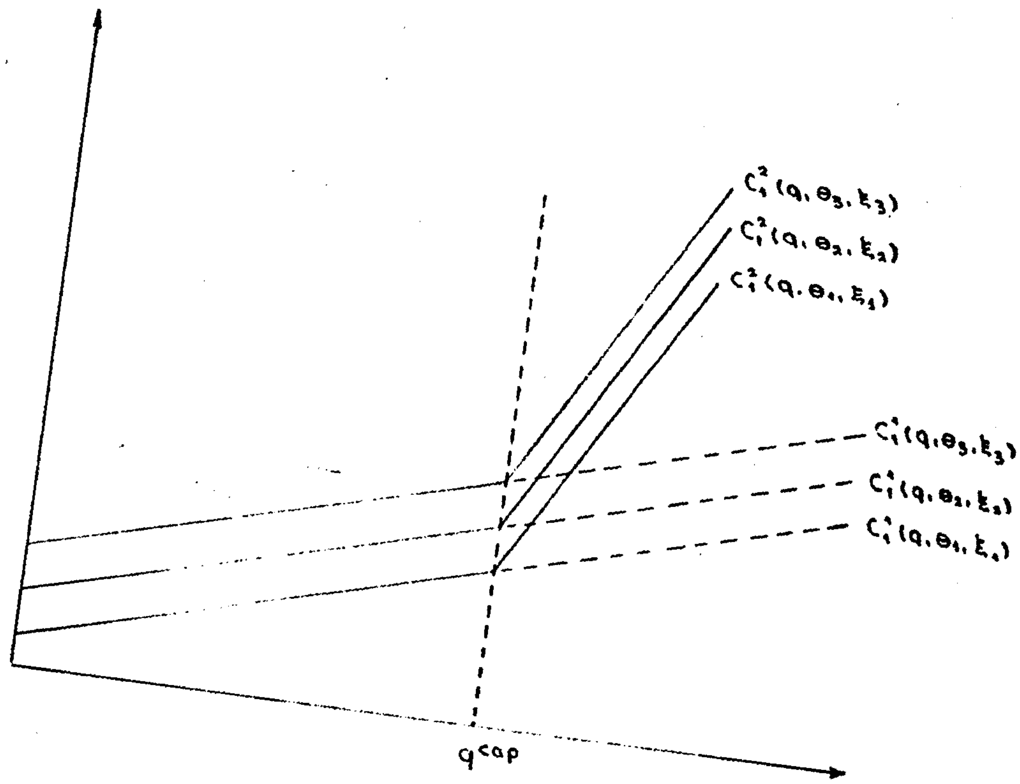
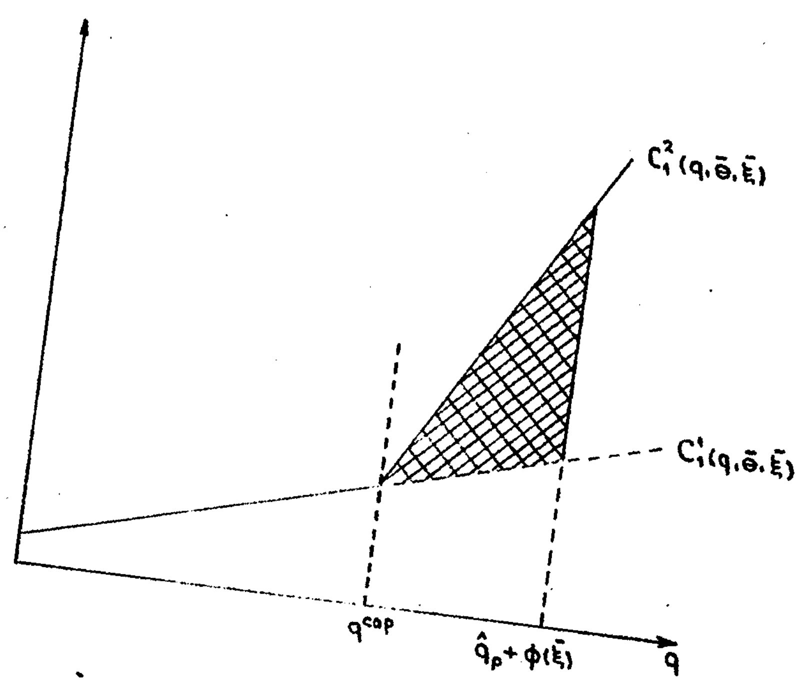


FIGURE 2.10

FIGURE 2.11



where $A > 0$.[26] We conclude, as expected, that the center no longer fully adapts to the output distortion of quantity orders.

The effect on the comparative advantage of prices is also easily deduced. If there exist states of nature such that the optimal price order intersects the steeper cost schedule, then there is a positive bias toward prices, since output variation under prices is less, in these states, than it was without the capacity constraint. When we state this result, we are, of course, drawing on the fully developed influence of C_{11} in transforming cost disturbances into output variation under price regulation.

Section 2.3: Uncertainty Between Production and Consumption

The motivating example for this section is any air pollution case in which the amount of pollution being emitted from a smokestack (the quantity produced) is typically far greater than the amount being breathed by the immediate population (the quantity consumed). To model this example, we write

$$q_c = q_a + \psi(\lambda),$$

[26]The first order condition that determines the optimal quantity order is now

$$E[(C_{11}-B_{11})(\hat{q}_p+\phi(\xi)-\hat{q}_o) + (C_{11}^2-C_{11}^1)[\max(0;(\hat{q}_p+\phi(\xi)-q^{cap}))]] = 0$$

Under the constraint listed in the text

$$(C_{11}^1-C_{11}^2)[\max(0;[\hat{q}_p+\phi(\xi) - q^{cap}]) > 0.$$

Therefore, we can conclude that

$$\hat{q}_p = \hat{q}_o - E\phi - \left(\frac{C_{11}^2-C_{11}^1}{C_{11}-B_{11}}\right)E[\max(0;[\hat{q}_p + \phi(\xi) - q^{cap}])].$$

where q_c is the quantity consumed, q_a is the quantity actually produced, and $\psi(\lambda)$ is the distortion created by the vector of random variables λ. The joint distribution of the four random variables is now

$$f_{\theta\eta\xi\lambda}(\theta,\eta,\xi,\lambda).$$

Costs are, of course, a function of q_a, and benefits depend on q_c.

For any quantity order issued by the center, $\bar{q}_p$, and arbitrary values of $\bar{\theta}$, $\bar{\xi}$, $\bar{\eta}$, and λ, the efficiency loss is

$$L(\bar{q}_p;\bar{\theta},\bar{\xi},\bar{\eta},\bar{\lambda}) = -\int_{\bar{q}_p+\phi(\bar{\xi})+\psi(\bar{\lambda})}^{q^{opt}+\psi(\bar{\lambda})} B_1(q,\bar{\eta})dq + \int_{\bar{q}_p + \phi(\bar{\xi})}^{q^{opt}} C_1(q,\bar{\theta},\bar{\xi})dq.$$

The center selects the optimal quantity order by minimizing expected losses and must therefore solve the following first order condition:

$$0 = -B_{11}E(\hat{q}_p + \phi(\xi) + \psi(\lambda) - \hat{q}_o) + C_{11}E(\hat{q}_p + \phi(\xi) - \hat{q}_o)$$

The optimal quantity order, then, is given by

$$\hat{q}_p = \hat{q}_o - E\phi - \frac{B_{11}E\psi}{B_{11}-C_{11}} . \qquad (2.3.1)$$

The corresponding optimal price order can be seen to equal

$$\tilde{p} = C' - (\frac{C_{11}B_{11}E\psi}{(B_{11}-C_{11})}).$$

by noting that the "effective" reaction function of the firm to any price order is[27]

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[27]By "effective response function," we mean the quantity actually consumed, given $\psi(\lambda)$, after the firm reacts to $\tilde{p}$, θ, and ξ in deciding to produce the profit maximizing amount.

$$q(p,\theta,\xi,\lambda) = \hat{q}_o + \frac{p-C'-\alpha(\theta,\xi)}{C_{11}} + \psi(\lambda).$$

The production response function to the optimal price is

$$\tilde{q}(\theta,\xi) = \hat{q}_o - \frac{\alpha(\theta,\xi)}{C_{11}} - \frac{B_{11}\ E\psi}{B_{11}-C_{11}} . \qquad (2.3.2)$$

The comparative advantage of prices over quantities is deduced from (2.3.1) and (2.3.2):

$$\Delta_5 = -E \left\{ \int_{\tilde{q}(\theta,\xi)+\psi(\lambda)}^{\hat{q}_p+\phi(\xi)+\psi(\lambda)} B_1(q,\eta)dq - \int_{\tilde{q}(\theta,\xi)}^{\hat{q}_p+\phi(\xi)} C_1(q,\theta,\xi)dq \right\}$$

$$= -E \left\{ \int_{\tilde{q}(\theta,\xi)+\psi(\lambda)}^{\tilde{q}(\theta,\xi)} B_1(q,\eta)dq + \int_{\tilde{q}(\theta,\xi)}^{\hat{q}_p+\phi(\xi)} (B_1(q,\eta)-C_1(q,\theta,\xi))dq + \int_{\hat{q}_p+\phi(\xi)}^{\hat{q}_p+\phi(\xi)+\psi(\lambda)} B_1(q,\eta)dq \right\}$$

Observe initially that

$$E \int_{\tilde{q}(\theta,\xi)+\psi(\lambda)}^{\tilde{q}(\theta,\xi)} (B_1)dq = -\frac{1}{2}B_{11}\left(E\psi^2 + 2E\left(\psi\cdot\left(\frac{-\alpha}{C_{11}}\right)\right) + \frac{B_{11}(E\psi)^2}{B_{11}-C_{11}}\right) - E(\beta(\eta)\cdot\psi(\lambda));$$

$$E \int_{\hat{q}_p+\phi(\xi)}^{\hat{q}_p+\phi(\xi)+\psi(\lambda)} (B_1)dq = \frac{1}{2} B_{11}\left(E\psi^2+2E(\psi\cdot\phi)-2E\psi E\phi+ \frac{B_{11}(E\psi)^2}{B_{11}-C_{11}}\right) + E(\beta\cdot\psi).$$

Therefore, we conclude that

$$\Delta_5 = - B_{11}(\mathrm{Cov}(\phi;\psi) - \mathrm{Cov}(\frac{-\alpha}{C_{11}} ; \psi) + \Delta_3, \qquad (2.3.3)$$

where Δ_3 is defined in subsection 2.2.1. The two covariances reflect the correlation between the potential sources of output variation and the consumption distortion.[28] If Cov ($\phi;\psi$) is positive, for example, the consumption distortion and the output distortion under quantities tend to move in the same direction, amplifying each other. Such an increase in variance is harmful, and we note a positive bias toward price controls. Since these effects enter the model only through the benefit function, B_{11} is the lone coefficient. It is also important to note that, aside from these correlation effects, the consumption distortion is neutral.

The reasons for this neutrality are twofold; both are revealed by the following theorem:

Theorem 1:

> Given a quadratic valuation function and two disturbances around identical means, the relative expected valuation of the two disturbances is invariant under arbitrary translation.

Notice that our present example easily satisfies the conditional clause of the theorem. Our valuation function is simply the benefit function, since the consumption distortion affects only the quantity consumed. The benefit function is indeed quadratic. Observe further that

$$E\hat{q}_c = E\,\tilde{q}\,(\theta,\xi) = \hat{q}_o - \left(\frac{B_{11}\,E\psi}{B_{11}-C_{11}}\right),$$

[28] The correlation effect between $\psi(\lambda)$ and the benefit function influences both modes equally, and therefore cancels.

and that since $\psi(\lambda)$ is additive, we are effectively adding $E(\psi(\lambda))$ to both $\hat{q}_c$ and $\tilde{q}(\theta,\xi)$ for all (θ,ξ). Theorem 1 therefore reestablishes the conclusion of (2.3.3) when λ is independent of θ, ξ, and η.

To prove the theorem, let

$$V(x) = V_o + V_1(x) + V_{11}(x)^2 \tag{2.3.4}$$

be the valuation function and $d_1(x)$ and $d_2(x)$ be the two disturbances around a single mean x_o. Assume that $f_x(x)$ distributes x and that there exists a subset S of the domain of $f_x(x)$ such that

$$d_1(x) \neq d_2(x) \text{ for any } x\ S\text{, and}$$

$$\int_{x\epsilon S} xf_x(x)dx \neq 0.$$

Rewrite (2.3.4) in the following form:

$$V(x) \ = \ v_o + v_1(x - x_o) + v_{11}(x - x_o)^2,$$

where

$$v_1 = V_1 + 2V_{11}x_o,$$

$$v_{11} = V_{11}\text{, and}$$

$$v_o = V_o + V_1x_o + V_{11}(x_o)^2.$$

The expected value of the first disturbance before translation is then

$$E(V(d_1(x))) = v_o + v_1Ed_1 + v_{11}E(d_1)^2$$

$$= v_o + v_1Ed_1 + v_{11}(Var(d_1) - (Ed_1)^2)$$

Similarly, for the second disturbance,

$$E(V(d_2(x))) = v_o + v_1Ed_2 + v_{11}(Var(d_2) - (Ed_2)^2)$$

We define the "relative expected valuation" as the difference of these two expressions and thus,

$$E(V(d_1)-V(d_2)) = v_1(Ed_1-Ed_2) + v_{11}(Var(d_1)-Var(d_2)) + v_{11}((Ed_1)^2 - (Ed_2)^2).$$

Suppose, for the sake of argument, that we translate the first disturbance by an amount L_1 and the second by an amount L_2. Then,

$$E(V(d_1+L_1)) = v_o + v_1(Ed_1+L_1) + v_{11}(Var(d_1)) + v_{11}((Ed_1)^2 + 2L_1Ed_1 + (L_1)^2).$$

There is a similar expression for $E(V(d_2+L_2))$ so that the relative valuation after translation is given by

$$v_1(Ed_1-Ed_2) + v_{11}(Var(d_1)-Var(d_2)) + v_{11}((Ed_1)^2-(Ed_2)^2) + v_{11}(L_1^2-L_2^2) + v_{11}(2L_1Ed_1-2L_2Ed_2) + v_1(L_1-L_2).$$

The relative valuation will remain constant if $Ed_1 = Ed_2$ and $L_1 = L_2$. We also note, for future reference, that when $Ed_1 \neq Ed_2$, the relative valuation is altered by $2v_{11}L_1(Ed_1-Ed_2)$, even if $L_1 = L_2$. It was therefore crucial, for the validity of our neutrality result, that both output distortions have the same mean.

It is also instructive to note the necessity of the quadratic valuation function in our result. The relative expected valuation will always be altered by translation if the value function is nontrivially cubic, even when the disturbances have equal means and are translated equal distances. To see this, observe that in the cubic case when $Ed_1 = Ed_2$, the relative expected valuation is

$$v_{11}(Var(d_1)-Var(d_2)) + 3v_{111}(Ed)(Var(d_1)-Var(d_2))$$

before translation, and

$$v_{11}(Var(d_1)-Var(d_2)) + 3v_{111}(Ed+L)(Var(d_1)-Var(d_2))$$

after translation (of an arbitrary length L). The difference of these expressions is the crucial statistic; it is

$$3v_{111}L(Var(d_1)-Var(d_2)), \text{ i.e.,} \qquad (2.3.5)$$

the change in the slope of the marginal value schedule between $(x_o + Ed)$ and $(x_o + Ed + L)$ times the difference in the variance of the two disturbances.

The relative effects of the variances of the two disturbances on the expected valuation are therefore increased (decreased) if the translation moves the disturbances into a region in which the value function is more (less) highly curved.[29] Returning to our model, if $E\psi(\lambda) < 0$, for example, then the mode of control that allows the greater variance in output is shackled with an even greater negative bias in the comparative advantage of prices, when $B_{111} > 0$. Notice, however, that (2.3.5)

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[29]Note that if $v_{111} > 0$, $L < 0$ moves the disturbances into a more highly curved region. Alternatively, if $v_{111} > 0$ and $L > 0$, they are moved into a flatter region.

guarantees that this third order effect of the consumption distortion only accentuates or dampens the overall effect of output variation.[30] We have, nonetheless, discovered a significant economic influence that was submerged by the symmetry of our quadratic approximations.

Section 2.4: The Impact of Inaccurate Information

We have required, in the previous sections of this chapter, that the center issue either a price order, or a quantity order, before the values of the random variables that influence the production outcomes of those orders become known. The center has been equipped, thus far, with precise knowledge of the distribution of the random variables. It is quite likely, however, that the center is not so equipped. The subjective distribution with which it makes its decisions will be inaccurate for one, or many, of a variety of reasons (an unperceived bias in measurement, an insufficiently fine measurement grid, etc.). In this section, we require that the center issue orders on the basis of expected value computations performed with an inaccurate distribution ($\hat{f}_{\theta\xi\eta\lambda}(\theta,\xi,\eta,\lambda) \neq f_{\theta\xi\eta\lambda}(\theta,\xi,\eta,\lambda)$ for example), and ask what effect, if any, this complication has on the comparative advantage of prices in the cases that we have just studied.

2.4.1: Uncertainty under Quantity Control

For any quantity order, $\bar{q}_p$, and arbitrary values of $\bar{\theta}$, $\bar{\xi}$, and $\bar{\eta}$,

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[30]We are discussing only the variation effect. If, without the consumption distortion, prices were slightly preferred, even though $Var(\frac{-\alpha}{C_{11}}) > Var(\phi)$, the consumption distortion could reverse the overall prices-quantities preference.

there exists an efficiency loss given by

$$L(\bar{q}_p;\bar{\theta},\bar{\xi},\bar{\eta}) = -\int_{\bar{q}_p + \phi(\bar{\xi})}^{q^{opt}} (B_1(q,\bar{\eta}) - C_1(q,\bar{\theta},\bar{\xi}))dq.$$

The center selects the optimal quantity order, $\hat{q}_p$, by minimizing the subjective expected value of these losses,

$$\hat{E}(L(\bar{q}_p;\ \bar{\theta},\ \bar{\xi},\ \bar{\eta})),$$

where $\hat{E}(---)$ represents the integral operator

$$\int_\theta \int_\xi \int_\eta (---)\ \hat{f}_{\theta\xi\eta}(\theta,\xi,\eta)d\eta d\xi d\theta.$$

The first order condition of this minimization determines

$$\hat{q}_p = \hat{q}_o - \hat{E}\phi(\xi) + \left(\frac{\hat{E}\alpha(\theta,\xi)-\hat{E}\beta(\eta)}{(B_{11}-C_{11})}\right). \tag{2.4.1}$$

The reaction function of the firm for any price, p, is

$$q(p,\theta,\xi) = \hat{q}_o + \frac{p-c'-\alpha(\theta,\xi)}{C_{11}} \equiv h(p,\theta,\xi), \tag{2.4.2}$$

so that

$$h_1\ (p,\theta,\xi) = (1/C_{11}).$$

The center knows the reaction function, and the optimal price order is computed through the following sequence of equations:

$$\tilde{p} = \frac{\hat{E}(B_1(h,n)\cdot h_1}{\hat{E}h_1}$$

$$= \hat{E}B_1(h(\tilde{p},\theta,\xi),n)$$

$$= B' + (\frac{B_{11}}{C_{11}})(\tilde{p} - C' - \hat{E}\alpha) + \hat{E}\beta.$$

Combining terms,

$$(1 - \frac{B_{11}}{C_{11}})\tilde{p} = (1 - \frac{B_{11}}{C_{11}})\ C' - \frac{B_{11}}{C_{11}}\hat{E}\alpha + \hat{E}\beta,$$

and finally,

$$\tilde{p} = C' + \frac{B_{11}\ \hat{E}\alpha}{B_{11}-C_{11}} - \frac{C_{11}\ \hat{E}\beta}{B_{11}-C_{11}} \ .$$

The quantity response to this order is

$$\tilde{q}(\theta,\xi) = \hat{q}_o - \frac{\alpha(\theta,\xi)}{C_{11}} + (\frac{B_{11}}{C_{11}})(\frac{\hat{E}\alpha}{B_{11}-C_{11}}) - (\frac{\hat{E}\beta}{B_{11}-C_{11}}). \quad (2.4.3)$$

Defining

$$\hat{Q} \equiv (\frac{\hat{E}\alpha}{B_{11}-C_{11}}), \text{ and } \hat{Q}' \equiv (\frac{\hat{E}\beta}{B_{11}-C_{11}}),$$

we can rewrite (2.4.1) and (2.4.3) in more manageable form:

$$\hat{q}_p = \hat{q}_o - E(\phi) + \hat{Q} - \hat{Q}', \text{ and} \quad (2.4.1)'$$

$$\tilde{q}(\theta,\xi) = \hat{q}_o - (\frac{\alpha(\theta,\xi)}{C_{11}}) + (\frac{B_{11}}{C_{11}})\hat{Q} - \hat{Q}'. \quad (2.4.3)'$$

We are now capable of computing and interpreting the comparative advantage of prices under imperfect information.[31]

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[31]The "hat" notation over delta indicates that the center used the incorrect distribution f in making its decisions. The subscript refers the reader to the previous delta that was computed under the same behavioral conditions when the center had the correct distribution.

$$\hat{\Delta}_3 = -E \int_{\tilde{q}(\theta,\xi)}^{\hat{q}_p + \phi(\xi)} (B_1(q,\eta) - C_1(q,\theta,\xi))dq$$

$$= -E\left\{ \int_{\tilde{q}(\theta,\xi)}^{\hat{q}_o - \frac{\alpha}{C_{11}}} (B_1-C_1)dq + \int_{\hat{q}_o - \frac{\alpha}{C_{11}}}^{\hat{q}_o - E\phi+\phi(\xi)} (B_1-C_1)dq + \int_{\hat{q}_o-E\phi+\phi(\xi)}^{\hat{q}_p+\phi(\xi)} (B_1-C_1)dq \right\}. \qquad (2.4.4)$$

The second integral is familiar; it is Δ_3. Its solution is recorded by equation (2.2.4) and its interpretation is presented in great detail in the subsequent text. The first integral represents the loss caused by the discrepancy from the perfect information case in the output response to the price order. The firm is still able to react precisely to $\tilde{p}$, θ, and ξ in deciding its output, but the incorrect subjective probability distribution distorts the optimal price order by $(B_{11}\hat{Q} - C_{11}\hat{Q}')$. The last integral similarly represents the loss caused by inaccurate specification of the optimal quantity order.

Solving equation (2.4.4) explicitly, we see that

$$\Delta_3 = \frac{1}{2}(B_{11}-C_{11})E(\frac{B_{11}}{C_{11}}\hat{Q}-\hat{Q}')^2 + \Delta_2 + \frac{1}{2}(B_{11}-C_{11})(\hat{Q}-\hat{Q}'+E(\phi-E\phi))^2$$

$$= \frac{1}{2}(B_{11}-C_{11})(Var(\frac{-\alpha}{C_{11}})+(\frac{\hat{E}\alpha}{C_{11}})^2) + \frac{1}{2}(B_{11}-C_{11})(Var(\phi)+(E\phi-\hat{E}\phi)^2)$$

$$+ [Cov(\frac{-\alpha}{C_{11}};\beta)+(\hat{E}\beta\hat{E}(\frac{-\alpha}{C_{11}})] + [Cov(\alpha;\phi)-(\hat{E}\alpha)(E\phi-\hat{E}\phi)]$$

$$- [Cov(\beta;\phi) - (\hat{E}\beta)(E\phi-\hat{E}\phi)] \qquad (2.4.5)$$

Equation (2.4.5) is easily interpreted. Consider the second moment of the output variation allowed by prices around $(\frac{-\hat{E}\alpha}{C_{11}})$, the center's per-

ception of the mean:

the output variation allowed by prices around $(\frac{-E\alpha}{C_{11}})$, the center's per-

ception of the mean:

C_{11} C_{11} C_{11} C_{11}

Notice that this is precisely the first term of (2.4.5) without the coefficient. Similarly,

$$\text{Var}(\phi) + (E\phi - \hat{E}\phi)^2 = E(\phi^2) - (E\phi)^2 + (E\phi)^2 - 2E\phi\hat{E}\phi + (\hat{E}\phi)^2$$

$$= E(\phi - \hat{E}\phi)^2,$$

the second moment of ϕ around the center's view of its mean, $\hat{E}\phi$. The covariance terms can also be thought of as the covariance of two effects around incorrect means. For instance,

$$\text{Cov}(\beta;\phi) - (\hat{E}\beta(E\phi - \hat{E}\phi)) = E(\beta\cdot\phi) - \hat{E}\beta E\phi + \hat{E}\beta\hat{E}\phi$$

$$E(\beta\cdot\phi) - E\beta E\phi - \hat{E}\beta E\phi + \hat{E}\beta\hat{E}\phi$$

$$E(\beta - \hat{E}\beta)(\phi - \hat{E}\phi).$$

The incorrect subjective distribution has accomplished nothing but a change in the points around which the variation is measured. The entire analysis of variance that was fully developed in Subsection 2.2.1 can therefore be applied here, totally intact; we need only recall the change in the expansion points.

2.4.2: Output Uncertainty at the Firm

The first subsection discusses the largest gap in knowledge that can exist between the center and the firm. We modify that condition in this subsection and assume that the firm, as well as the center, cannot view the values of the relevant random variables before making its output decision. In addition, we introduce output uncertainty, still indexed by ξ, into the quantity orders issued at the firm (in response to a price order). The <u>ex post</u> reaction to the value of ξ that actually occurs in this context should be independent of the origin of the quantity order. The divergence in actual output from a quantity order issued at the plant level should therefore be equal to the divergence in output from a centrally issued command, for any ξ. The difference between actual output, q_a, and ordered output, q_p, is represented by $\phi(\xi)$ regardless of the source of the order. Allowing that the manager of the peripheral firm will have a different perception of ξ (finer measurement grid, more recent data, etc.), we further assume that the manager works with a subjective distribution, $\tilde{f} \neq \hat{f}$. Information need not be perfect, even at the firm level, so $\tilde{f} \neq f$, as well.

Computation of the optimal quantity order remains the same as before:

$$\hat{q}_p = \hat{q}_o - \hat{E}\phi + \hat{Q} - \hat{Q}'. \tag{2.4.6}$$

The price reaction function of the firm has changed markedly, however, and the computation of the optimal price reflects this change. The firm now maximizes expected profits, given any price order, p, to select its output order, $q_p(p)$. The first order condition for this maximization is

$$p = \tilde{E}(C_1(q_p(p) + \phi(\xi), \theta, \xi))$$

$$= C' + \tilde{E}\alpha + C_{11}(q_p(p) + \tilde{E}\phi - \hat{q}_o)$$

where $\tilde{E}(---)$ represents the expected value as computed by the firm:

$$\tilde{E}(---) = \int_{\theta}\int_{\xi}\int_{\eta} (---)\ \tilde{f}_{\theta\xi\eta}(\theta,\xi,\eta)d\eta d\xi d\theta .$$

The reaction function is therefore

$$q_p(p) = \hat{q}_o - \tilde{E}\phi + \frac{\tilde{p} - C' - \tilde{E}\alpha}{C_{11}} . \qquad (2.4.7)$$

The center, however, must deduce the firm's reaction function from its own information, and therefore believes that it is

$$q_p^e(p) = \hat{q}_o - \hat{E}\phi + \frac{\tilde{p} - C' - \hat{E}\alpha}{C_{11}} . \qquad (2.4.7)^e$$

As a result, it is equation $(2.4.7)^e$ that is used to compute the optimal price order:

$$\tilde{p} = \hat{E}(B_1(q_p^e(p) + \phi(\xi),\eta)$$

$$= B' + \hat{E}\beta + B_{11}\left(\frac{\tilde{p} - C' - \hat{E}\alpha}{C_{11}} + \hat{E}\phi - \hat{E}\phi\right),$$

so that

$$\tilde{p} = C' - C_{11}\hat{Q}' + B_{11}\hat{Q} .$$

The reaction to the optimal price order is now constant:

$$\tilde{q}_p = \hat{q}_o - \left(\frac{\tilde{E}\alpha}{C_{11}}\right) - \tilde{E}\phi + \left(\frac{B_{11}}{C_{11}}\right)\hat{Q} - \hat{Q}' . \qquad (2.4.8)$$

Notice that (2.4.8) was derived by inserting p into (2.4.7), not $(2.4.7)^e$.

The comparative advantage of prices over quantities is again available:

$$\Delta_6 = -E\int_{\tilde{q}_p + \phi(\xi)}^{\hat{q}_p + \phi(\xi)} (B_1-C_1)dq = -\frac{1}{2}(B_{11}-C_{11})[(E\phi-\hat{E}\phi+\hat{Q}-\hat{Q}')^2 - (E\phi-\tilde{E}\phi+\frac{B_{11}}{C_{11}}\hat{Q}-(\frac{\tilde{E}\alpha}{C_{11}})-\hat{Q}')^2]$$

Observe that since $(-\frac{1}{2})(B_{11}-C_{11}) > 0$,

$$\Delta_6 \gtrless 0 \iff (E\phi-\hat{E}\phi+\hat{Q}-\hat{Q}')^2 \gtrless (E\phi-\tilde{E}\phi+(\frac{B_{11}}{C_{11}})\hat{Q} - (\frac{\tilde{E}\alpha}{C_{11}}) - \hat{Q}')^2$$

$$\iff |E\phi - \hat{E}\phi + \hat{Q} - \hat{Q}'|$$

$$\gtrless |E\phi - \hat{E}\phi + (\frac{B_{11}}{C_{11}})\hat{Q} - (\frac{\tilde{E}\alpha}{C_{11}}) - \hat{Q}'|. \qquad (2.4.9)$$

With perfect information,[32] $\hat{q}_p = \hat{q}_o - E(\phi) = \tilde{q}_p$. The expression

$$|E(\phi) - \tilde{E}(\phi) + \hat{Q} - \hat{Q}'|$$

is therefore the absolute magnitude of the error made by the center in setting its quantity order;

$$|E(\phi) - \tilde{E}(\phi) + (\frac{B_{11}}{C_{11}})\hat{Q} - (\frac{\tilde{E}\alpha}{C_{11}}) - \hat{Q}'|$$

is the absolute magnitude of the error made by the peripheral firm in making its quantity decision. Equation (2.4.9) reveals that the decision maker who makes the smaller error in issuing a quantity order should be allowed to do so; i.e., if the absolute value of the error made by the center in a quantity decision is less than the error made at the periphery, the center should issue the quantity. Otherwise, the center should issue

[32]Perfect information here is loosely defined as perfect knowledge of the relevant distribution; i.e., $\hat{f} = \tilde{f} = f$.

a price order and the peripheral firm decides the intended output.

2.4.3: The Consumption Distortion with Errors

We now test the general validity of the results of Subsection 2.4.1 by reconsidering the random distortion between the quantity produced and the quantity consumed in the context of the model presented there: the peripheral firm is able to observe θ and ξ before making its output decision, while the center is burdened with an inaccurate subjective distribution in deciding its order.

Noting the standard efficiency loss for any quantity order, the center selects the optimal quantity order, $\hat{q}_p$, by minimizing the expected value of these losses. It solves, therefore, the first order condition that[33]

$$\hat{E}\, B_1\, (\hat{q}_p + \phi(\xi) + \psi(\lambda),\eta) - \hat{E}\, C_1\, (\hat{q}_p + \phi(\xi),\theta,\xi) = 0.$$

The optimal quantity order is thus

$$\hat{q}_p = \hat{q}_o - \hat{E}\phi + \left(\frac{\hat{E}\alpha - \hat{E}\beta}{B_{11}-C_{11}}\right) - \left(\frac{B_{11}\, \hat{E}\psi}{B_{11}-C_{11}}\right). \qquad (2.4.10)$$

Recall that $\hat{Q} \equiv \left(\frac{\hat{E}\alpha}{B_{11}-C_{11}}\right)$ and $\hat{Q}' \equiv \left(\frac{\hat{E}\beta}{B_{11}-C_{11}}\right)$, and define

$$Q'' \equiv \left(\frac{\hat{E}\psi}{B_{11}-C_{11}}\right).$$

Equation (2.4.10) can be written

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[33] Recall that $\psi(\lambda)$ is the random distortion between the quantity consumed and the quantity produced (see Section 2.3).

$$\hat{q}_p = \hat{q}_o - \hat{E}\phi + \hat{Q} - \hat{Q}' - B_{11}\hat{Q}''. \qquad (2.4.10)'$$

The reaction function of the firm is independent of $\psi(\lambda)$, so that

$$\tilde{p} = \hat{E}\, B_1\, (h(\tilde{p},\theta,\xi) + \psi(\lambda),\eta),$$

where $h(\tilde{p},\theta,\xi)$ is defined by equation (2.4.2). The optimal price order is therefore

$$\tilde{p} = C' + B_{11}\,\hat{Q} - C_{11}\,\hat{Q}' - B_{11}C_{11}\,\hat{Q}'', \qquad (2.4.11)$$

and the response curve for $\tilde{p}$ is

$$\tilde{q}(\theta,\xi) = \hat{q}_o - \frac{\alpha(\theta,\xi)}{C_{11}} + \left(\frac{B_{11}}{C_{11}}\right)\hat{Q} - \hat{Q}' - B_{11}\hat{Q}'' \qquad (2.4.12)$$

Equations (2.4.10) and (2.4.12) combine to generate the comparative advantage of prices:

$$\hat{\Delta}_5 = -E\int_{\tilde{q}(\theta,\xi)+\psi(\lambda)}^{\hat{q}_p+\phi(\xi)+\psi(\lambda)} B_1(q,\eta)\,dq + E\int_{q(\theta,\xi)}^{\hat{q}_p+\phi(\xi)} C_1(q,\theta,\xi)\,dq$$

$$= -E\left\{\int_{\tilde{q}(\theta,\xi)+\psi(\lambda)}^{\hat{q}_o-\frac{\alpha(\theta,\xi)}{C_{11}}+\frac{B_{11}}{C_{11}}\hat{Q}-\hat{Q}'} B_1(q,\eta)\,dq + \int_{\hat{q}_o-\frac{\alpha}{C_{11}}+\frac{B_{11}}{C_{11}}\hat{Q}-\hat{Q}'}^{\hat{q}_o-\hat{E}\phi+\hat{Q}-\hat{Q}'+\phi(\xi)} (B_1(q,\eta)-C_1(q,\theta,\xi))\,dq\right.$$

$$+\int_{\hat{q}_o-\hat{E}\phi+\hat{Q}-\hat{Q}'+\phi(\xi)}^{\hat{q}_p+\phi(\xi)} B_1(q,\eta)\,dq - \int_{\tilde{q}(\theta,\xi)}^{\hat{q}_o-\frac{\alpha}{C_{11}}+\frac{B_{11}}{C_{11}}\hat{Q}-\hat{Q}'} C_1(q,\theta,\xi)\,dq - \int_{\hat{q}_o-\frac{\alpha}{C_{11}}+\frac{B_{11}}{C_{11}}\hat{Q}-\hat{Q}'}^{\hat{q}_p+\phi(\xi)} C_1(q,\theta,\xi)\,dq.$$

The second integral is familiar; equation (2.4.5) records its solution. The subsequent text explains it fully. Manipulation of the other integrals reveals that

$$\hat{\Delta}_5 = \hat{\Delta}_3 - B_{11}[(\mathrm{Cov}(\phi;\psi) + (E\phi - \hat{E}\phi)(E\psi - \hat{E}\psi)$$

$$- \mathrm{Cov}(\frac{-\alpha}{C_{11}};\psi) - \hat{E}(\frac{-\alpha}{C_{11}})(E\psi - \hat{E}\psi)] \qquad (2.4.13)$$

Observe that

$$(\mathrm{Cov}(\phi;\psi) + (E\phi - \hat{E}\phi)(E\psi - \hat{E}\psi)$$

$$= (E(\phi\cdot\psi)-E\phi E\psi) + (E\phi E\psi - E\phi\hat{E}\psi - \hat{E}\phi E\psi + \hat{E}\phi\hat{E}\psi)$$

$$= E(\phi-\hat{E}\phi)\ (\psi-\hat{E}\psi).$$

Since $(E(\frac{-\alpha}{C_{11}})) = 0$, a similar result holds for $(\mathrm{Cov}(\frac{-\alpha}{C_{11}};\ )-(\hat{E}(\frac{-\alpha}{C_{11}})(E\psi-\hat{E}\psi))$. The arguments of the B_{11} coefficient are thus the covariances of the two possible output disturbances and the consumption distortion computed around what the center believes to be the means of these effects. The analytics of section 2.3 can therefore be applied directly to this example, keeping in mind that all variation is measured from the incorrect means. The spirit of subsection 2.4.1 is also preserved.

Recall that in section 2.3, the neutrality of the consumption distortion was noted under the assumption that λ be independent of the other random variables. We also observed that if the two potential output disturbances had different means, then there would be a secondary effect even with independence. In the context of the present example, we will read the value of that effect from (2.4.13), demonstrate the derivation of that value from the valuation function argument of Section 2.3, and provide a geometric description of its origin.

The relevant result from Section 2.3 reads as follows: for any valuation function of the form $V(x) = v_o + v_1(x-x_o) + v_{11}(x-x_o)^2$, the relative expected value of two random disturbances in x, $d_1(x)$ and $d_2(x)$ is altered by $2v_{11}L(Ed_1-Ed_2)$ when both disturbances are translated a distance L. Allow that q is x, $\hat{q}_o$ is x_o, and define

$$d_1(q) \equiv \tilde{q}(\theta,\xi), \text{ and}$$

$$d_2(q) \equiv \hat{q}_p + \phi(\xi).$$

Therefore, $Ed_1-Ed_2 = (\hat{E}(\frac{-\alpha}{C_{11}})-(E\phi-\hat{E}\phi))$. The output that is inserted into the benefit function is translated a distance of $(E\psi - B_{11}\hat{Q}'')$ under either mode of control by the introduction of the consumption distortion. The result of Section 2.3 predicts a change in the relative expected valuation of the two disturbances (i.e., the comparative advantage of prices over quantities) equal to

$$B_{11}(E\psi - B_{11}\hat{Q}'')(\hat{E}(\frac{-\alpha}{C_{11}}) - (E\phi - \hat{E}\phi))$$

$$= B_{11}((\frac{B_{11}}{B_{11}-C_{11}})(E\psi - \hat{E}\psi) - (\frac{C_{11}}{B_{11}-C_{11}})\ E\psi\ (\hat{E}(\frac{-\alpha}{C_{11}}) - (E\phi - \hat{E}\phi))$$

The output that is inserted into the cost function is similarly translated $(-B_{11}\hat{Q}'')$ under either mode. The change in the relative expected valuation for this translation is

$$C_{11}((\frac{B_{11}}{B_{11}-C_{11}})\ \hat{E}\psi\ (\hat{E}(\frac{-\alpha}{C_{11}}) - (E\phi - \hat{E}\phi))$$

The total change in the relative expected valuation from these two effects is

$$B_{11}(E\psi-\hat{E}\psi)(\hat{E}(\frac{-\alpha}{C_{11}}) - (E\phi-\hat{E}\phi)) \qquad (2.4.14)$$

This expression is precisely equal to $\hat{\Delta}_5-\hat{\Delta}_3$ under the assumption that λ is independent of θ, ξ, and η; it is therefore the change in the comparative advantage of prices that results entirely from the center's inaccurate knowledge of the distribution of the consumption distortion.

We can construct a geometric arena in which the genesis of this result becomes quite clear. Figure (2.12) illustrates the case in which $E\phi-\hat{E}\phi \equiv \hat{D} > 0$. We begin by assuming that $\hat{E}\alpha = 0$ and define

$$\hat{q}_{ao} \equiv (\hat{q}_p + \phi(\xi) - \hat{q}_o)\Big|_{\substack{\hat{D} = 0 \\ \psi(\lambda) = E\psi.}}$$

The increase in benefits that is created by $D > 0$ when $\psi(\lambda)$ is held at $E\psi$ is

$$(B' = \beta(\eta))\hat{D} + \frac{1}{2} B_{11} (2\hat{D}\,\hat{q}_{ao} + \hat{D}^2).$$

The corresponding gain in benefits, however, when $\psi(\lambda)-E\psi < 0$ is

$$(B' + \beta(\eta))\hat{D} + \frac{1}{2} B_{11} (2\hat{q}_{ao}\hat{D} + 2\hat{D}(\psi(\lambda) - \hat{E}\psi) + \hat{D}^2)$$

so that the extra gain in benefits is

$$B_{11}\hat{D}(\psi(\lambda)-\hat{E}\psi).$$

This extra gain exists because $(\psi(\lambda)-\hat{E}\psi) < 0$ pushes the quantity consumed into a region in which the benefit function is more steeply sloped (see Figure (2.12)). Note that $B_{11}\hat{D}(\psi(\lambda)-\hat{E}\psi)$ is the extra loss that results when $\hat{D} < 0$ from the same steeper slope.

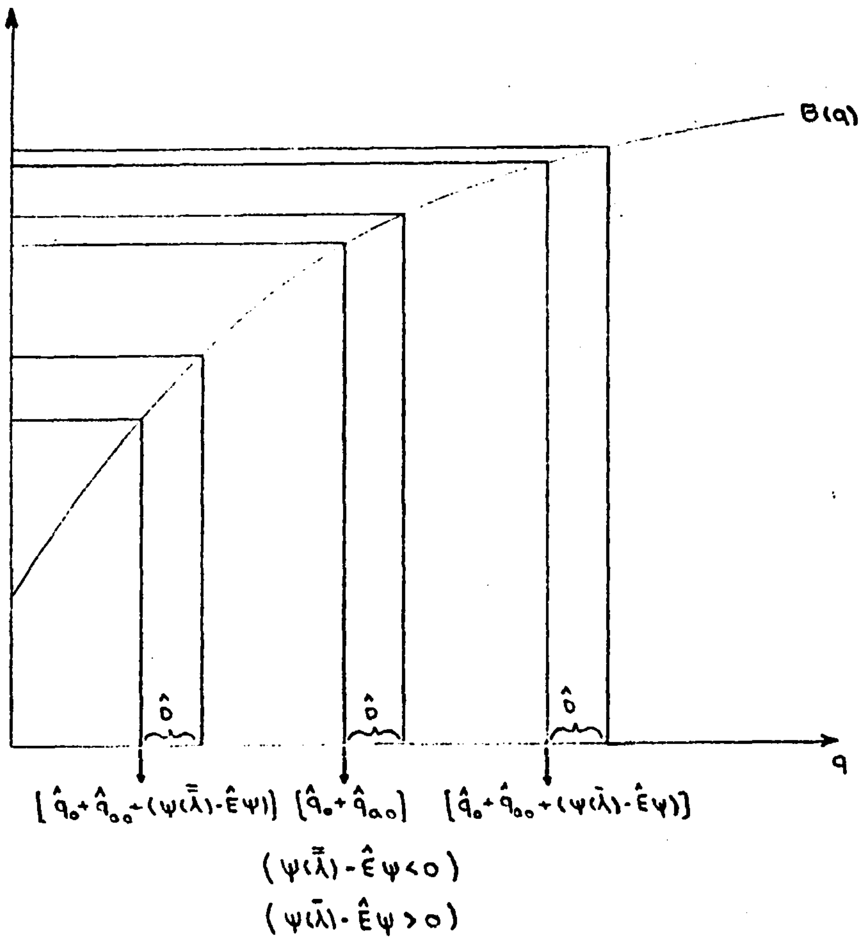

FIGURE 2.12

The quantity consumed enters the benefit function in a flatter region when, on the other hand, $\psi(\lambda)-\hat{E}\psi > 0$. We observe through parallel reasoning that

i) $B_{11}\hat{D}(\psi(\lambda)-\hat{E}\psi)$ is the loss in increased benefits over the case in which $\psi(\lambda) = \hat{E}\psi$ when $\hat{D} > 0$ rather than $\hat{D} = 0$.

ii) $B_{11}\hat{D}(\psi(\lambda)-\hat{E}\psi)$ is the decrease in lost benefits over the same case when $\hat{D} < 0$ instead of $\hat{D} = 0$.

Since the optimal price order is independent of $\hat{D}$, the above are the only effects when $\hat{E}\alpha = 0$, and reflect changes under quantity control only. The expression $B_{11}\hat{D}(E\psi-\hat{E}\psi)$ is therefore added to the expected benefits achieved under quantity control, or equivalently subtracted from the comparative advantage of prices.[34]

Similar reasoning can be employed to explain the appearance of $B_{11}(\hat{E}(\frac{-\alpha}{C_{11}}))(E\psi-\hat{E}\psi)$ in $\hat{\Delta}_5$. The crucial difference here is that the error

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[34]The following table shows that addition is the proper operation in all of the permutations of the signs of $\hat{D}$ and $(E\psi-\hat{E}\psi)$:

$(E\psi-\hat{E}\psi) < 0;\hat{D} > 0$	$\Rightarrow$	gain in benefits under quantities	$\Rightarrow$	$B_{11}(E\psi-\hat{E}\psi)\hat{D} > 0$ <u>added</u>
$(E\psi-\hat{E}\psi) < 0;\hat{D} < 0$	$\Rightarrow$	loss in benefits under quantities	$\Rightarrow$	$B_{11}(E\psi-\hat{E}\psi)\hat{D} < 0$ <u>added</u>
$(E\psi-\hat{E}\psi) > 0;\hat{D} > 0$	$\Rightarrow$	loss in benefits under quantities	$\Rightarrow$	$B_{11}(E\psi-\hat{E}\psi)\hat{D} < 0$ <u>added</u>
$(E\psi-\hat{E}\psi) > 0;\hat{D} < 0$	$\Rightarrow$	gain in benefits under quantities	$\Rightarrow$	$B_{11}(E\psi-\hat{E}\psi)\hat{D} > 0$ <u>added</u>

made by the center in evaluating the mean of marginal costs influences not only the quantity order, but also the price order. We begin as before, defining

$$\hat{q}'_{ao} \equiv (\hat{q}_p + \phi(\xi) - \hat{q}_o)\Big|_{\substack{\hat{E}\alpha = 0 \\ \psi(\lambda)=\hat{E}\psi}} \quad \text{and} \quad \tilde{q}_o(\theta,\xi) = (\tilde{q}(\theta,\xi) - \hat{q}_o)\Big|_{\substack{\hat{E}\alpha = 0 \\ \psi(\lambda)=\hat{E}\psi}}$$

and assuming that $\hat{D} = E\phi - \hat{E}\phi = 0$.[35] If $\hat{E}\alpha > 0$, the decrease in benefits under quantity control over the case where $\hat{E}\alpha = 0$ is

$$(B' + \beta(\eta))\,\hat{Q} + \frac{1}{2} B_{11}\,(2\hat{q}'_{ao}\,\hat{Q} + \hat{Q}^2) \qquad (2.4.14)$$

this very same expression represents the increase in benefits when E becomes negative. The term

$$(B' + \beta(\eta))\,(\frac{B_{11}}{C_{11}}\hat{Q}) + \frac{1}{2} B_{11}\,(2\tilde{q}_o(\theta,\xi)(\frac{B_{11}}{C_{11}}\hat{Q}) + (\frac{B_{11}}{C_{11}}\hat{Q})^2).$$

similarly represents the increase in benefits under price control when $\hat{E}\alpha > 0$ instead of $\hat{E}\alpha = 0$, and the decrease in benefits when $\hat{E}\alpha$ is negative.

The additional loss in benefits under quantity control for the case in which $\hat{E}\alpha > 0$ over the case in which $\hat{E}\alpha = 0$ is, however,

$$(B' + \beta(\eta))\,\hat{Q} + \frac{1}{2} B_{11}\,(2\hat{q}'_{ao}\,\hat{Q} + 2(\psi(\lambda)-\hat{E}\psi)\hat{Q} + \hat{Q}^2) \qquad (2.4.15)$$

when $\psi(\lambda)-\hat{E}\psi < 0$. We see by comparing (2.4.14) and (2.4.15) that

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[35]Treating each error individually by assuming the other is equal to zero is permissible since their effects are entirely separable.

$$B_{11}(\psi(\lambda)-\hat{E}\psi)\hat{Q}$$

is the extra loss incurred under quantity control when $\psi(\lambda)-\hat{E}\psi < 0$ and pushes the quantity consumed into a more steeply sloped region of the benefit function. The extra gain under price control is similarly

$$B_{11}(\psi(\lambda)-\hat{E}\psi)(\frac{B_{11}}{C_{11}})\hat{Q}.$$

As a result, $B_{11}(E\psi-\hat{E}\psi)\hat{Q}$ should be added to the expected loss under quantities and $B_{11}(E\psi-\hat{E}\psi)(\frac{B_{11}}{C_{11}})\hat{Q}$ should be subtracted from the expected gain under prices;

$$B_{11}(E\psi-\hat{E}\psi)\hat{Q} - B_{11}(E\psi-\hat{E}\psi)(\frac{B_{11}}{C_{11}})\hat{Q} = B_{11}(E\psi-\hat{E}\psi)(\frac{\hat{E}\alpha}{C_{11}})$$

should therefore be added to the comparative advantage of prices (summing the two effects). Perfectly analogous arguments produce the same result in the other three permutations of signs.

We have demonstrated that the consumption distortion moves the quantity consumed around in the domain of the benefit function. Since the means of the two possible output disturbances are different, the effect of this movement is asymmetric and the concavity of the benefit function generates

$$B_{11}(E\psi-\hat{E}\psi)(\hat{E}(\frac{-\alpha}{C_{11}}) - (E\phi-\hat{E}\phi))$$

in the expression for the comparative advantage of prices over quantities.

Section 2.5: An Example--Automobile Emissions

The proposed control of automobile emissions will provide an arena in which we can apply our analysis. Carbon monoxide standards will be studied in this chapter and the next; nitrous oxides will be included later in Chapter Four. These applications will be quite casual in nature, serving primarily as an illustration of the potential value of our results. We will, however, draw upon the empirical work of other researchers to provide this illustration with a bridge to reality.

Our present results reveal that the curvatures of the benefit and cost functions in the neighborhood of the desired emission levels (B_{11} and C_{11}) are of crucial importance. Benefit functions for the reduction of vehicular carbon monoxide and nitrous oxide emissions have been computed recently by William Ahern, Jr.[36] From a variety of sources, he has expressed benefits, in terms of "equivalent days of restricted activity" (EDRA) per year, as a function of the percentage reduction over the CO and NO_x emissions levels of a typical 1967 automobile. We will use his upper estimate of the average cost of an EDRA,[37] $11.00,[38] to express these benefits in dollars per year. We hope, in that way, to capture not only the human health effects recorded by an EDRA, but also at least part of the other pollution related damages.[39] Table (2.2)

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[36] William Ahern, Jr., "Measuring the Value of Emissions Reductions," in Jacoby and Steinbruner, Cleaning the Air, pp. 175-205.

[37] This is the amount that people are willing to pay to avoid an EDRA.

[38] Ibid., p. 202.

[39] Non-human biological effects on plants and wildlife, property damage, etc.

Table 2.2

EDRA Estimates for Carbon Monoxide

1967 Levels	50% Reduction	75% Reduction
1.42×10^9	2.35×10^8	1.90×10^7

Source: William Ahern, Jr., "Measuring the Value of Emissions Reductions," in Jacoby and Steinbruner, Cleaning the Air, p. 198.

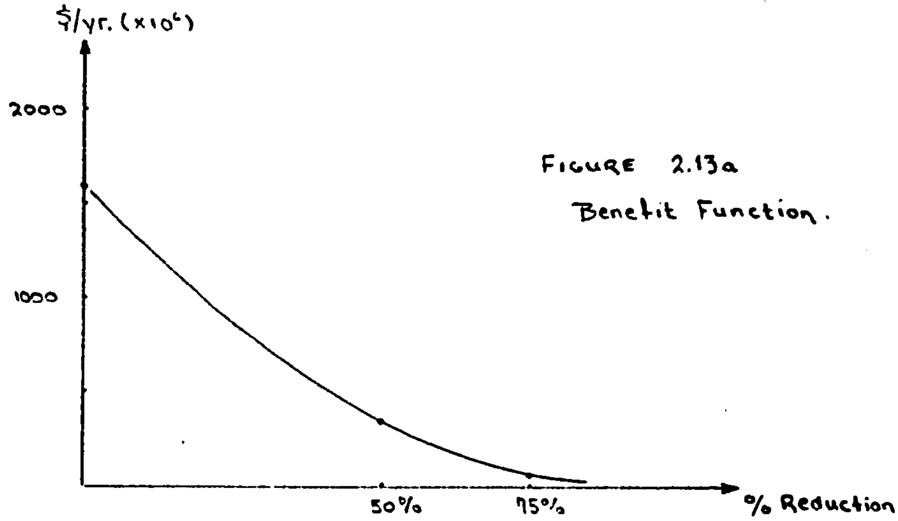

FIGURE 2.13a
Benefit Function.

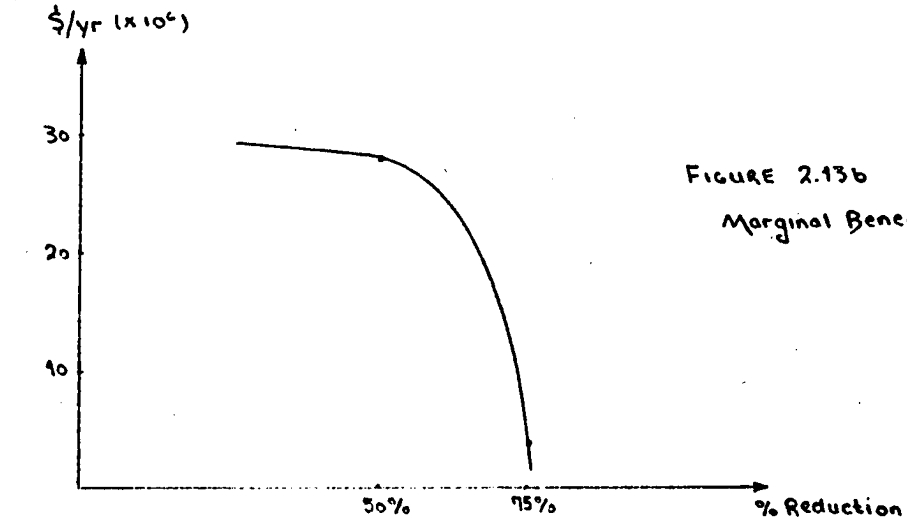

FIGURE 2.13b
Marginal Benefits

and Figures (2.13) summarize Ahern's findings for carbon monoxide. The curvature of benefits can then be approximated geometrically by estimating the slope of marginal benefits in Figure (2.13b).

Donald Dewes has calculated marginal cost schedules for the reduction of carbon monoxide and nitrous oxides as a function of the percentage reduction of the typical emissions of a 1963 automobile.[40] Since no pollution devices were required nationally between 1963 and 1967, the emissions of a typical 1963 automobile should nearly equal those of a 1967 vehicle. We will assume exact equality of these base levels and summarize the Dewes results for carbon monoxide under that assumption in Table (2.3)[41] and Figure (2.14).[42] The slope of this marginal schedule will give the relevant curvature for the cost function.

The variances of emissions under both modes of control are, of course, also essential. Recall that the center must issue either a single, binding price or quantity order before actual costs are known. The level of emissions produced under price control would therefore be unknown to the center until the actual production costs are perceived. To explore this output uncertainty before making its decision, the center could collect a set of cost estimates for each of the systems listed in Table (2.3). The expected cost and standard deviation for

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[40] Donald Dewes, Economics and Public Policy: The Automobile Pollution Case, MIT Press.

[41] Ibid., Appendix C.

[42] Dewes lists his data in dollars per mile, while marginal benefits are recorded in dollars per year. To correct this units discrepancy, we extrapolate the total mileage figures for the last few years as reported by the automobile manufacturers and suppose, quite reasonably, that 1.4 trillion vehicle miles will be driven in 1975.

Table 2.3

The Cost Side for CO

System (A): Positive Crankcase Ventilation

System (B): 1968 Clean Air Package plus (A)

System (C): 1970 Controlled Combustion System plus (A)

System (D): Low-Lead, Low Octane Engine plus 1971 Catalytic Exhaust Converter plus (C)

System	Emission	% Reduction	Change in Total Cost	Marginal Cost
No Control	76.7	---	----	----
(A)	76.7	---	.00027	----
(B)	38.3	50%	.00022	$.4 \times 10^{-5}$
(C)	25.0	69%	.00008	$.4 \times 10^{-5}$
(D)	2.3 (gm./mi.)	97%	.00688 ($/mi.)	25×10^{-5} $/(gm./mi.)

Source: Donald Dewes, Economics and Public Policy: The Automobile Pollution Case, MIT Press, Cambridge, 1974, Appendix C.

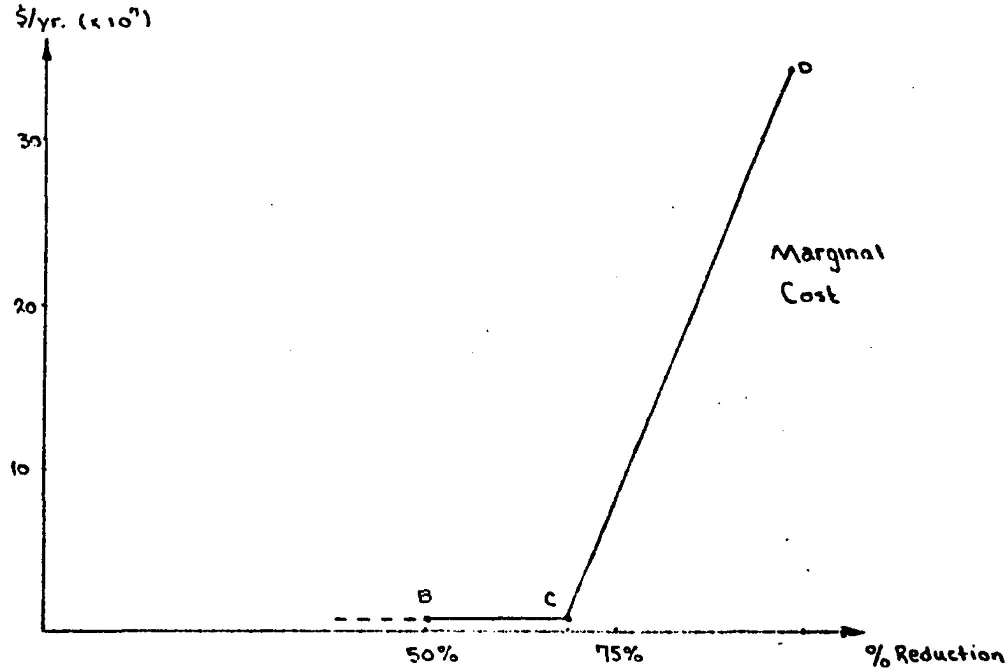

FIGURE 2.14

each system could then be computed and used to quantify the cost/profit motivated uncertainty that the center faced.

We can illustrate this procedure with the data collected by Dewes. Each cost figure that he lists for a particular system has been derived from a set of estimates provided by the Environmental Protection Agency, the National Academy of Sciences and the automobile manufacturers themselves. The range of these estimates for each system will suggest standard deviations in marginal costs similar to the statistics that the center would compute. If costs are quadratic, these standard deviations in marginal cost can be translated into standard deviations in emissions by dividing by the curvature of the cost function.[43] This computation will provide a rough notion of the profit motivated variance in vehicular emissions that the center would confront in this particular problem.

The total costs of system (D) for the manufacturer are, for example, given by Dewes to be $\$4.3 \times 10^9$/yr. The estimates of the EPA, the National Academy of Sciences, and the manufacturers suggest a standard deviation around that value of $\$5.4 \times 10^8$/yr.[44] The marginal cost of system (D) over system (C) is then $\$3.5 \times 10^8 \pm \$1.9 \times 10^7$ per year.[45] From the point of view of the center, then, this cost uncertainty implies

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[43]When costs are quadratic, disturbances in marginal costs and disturbances in output are related by a factor of $(1/C_{11})$, the inverse of the curvature of the cost function. As will become clear in the examples, this is a system by system computation.

[44]Ibid., p. 191.

[45]System (D) produces an additional 28% reduction in carbon monoxide emissions over system (C). $\$5.4 \times 10^8$/yr. divided by this 28% yields $\$1.9 \times 10^7$/yr.

a profit motivated standard deviation in emissions around the 97% reduction expected of system (D) equal to $\$1.9 \times 10^7$/yr. divided by the curvature of the yearly cost function in that neighborhood. Between C and D, that curvature is $\$1.3 \times 10^7$/yr. per 1% reduction, so that system (D) achieves its 97% reduction plus or minus 1.4%. The center must therefore work with a profit motivated variance in carbon monoxide emissions of

$$(1.4(76.7) \times 10^{-2})^2(\text{gm./mi.})^2 \; = \; 1.1 \; (\text{gm./mi.})^2$$

if it tries to achieve a 97% reduction in CO by price controls.

A similar computation for system (C) reveals that the total cost to the manufacturer is $\$5.5 \times 10^8$/yr. plus or minus $\$1.4 \times 10^8$/yr. for a 69% reduction. This cost uncertainty implies a profit induced variance in carbon monoxide emissions under prices of .3 $(\text{gm./mi.})^2$ from the center's point of view.

An automobile manufacturer must also deal with production variability in its emissions reducing devices as he responds to a control of either mode. This variability produces a second source of ex ante uncertainty with which the center must deal as it makes its control choice. Carbon monoxide emissions from automobiles subject to the 1975 quantity standards have exhibited a standard deviation of 20% of the required 3.4 gm./mi., for example.[46] The variance in CO output can

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[46]Milton Weinstein and Ian Clark, "Emissions Measurements and Testing of New Vehicles," in Jacoby and Steinbruner, Cleaning the Air, p. 106.

therefore be estimated at

$$(.2(3.4))^2(\text{gm./mi.})^2 \;=\; .4\ (\text{gm./mi.})$$

in this case under this quantity control. Such variability in output is directly attributed to the conscious quality control decisions of the automobile manufacturers. The center must therefore determine whether this variance in output would be set at the same level under an equivalent taxation scheme. To answer this question, the manufacturer's response to a quantity standard must be studied in greater detail. We will again attempt to illustrate the center's assessment procedures with the limited data that we have available.

An automobile manufacturer faces not only a quantity standard, but also the uncertainty of the testing procedures used to compare his product to that standard. Production variability as well as measurement error therefore contruct an output distribution around his expected observed emission level, e. He would never set his e equal to the given standard because such policy would imply a 50% chance of exceeding that standard. He would obviously pick a lower mean so that he meets the requirement x% of the time, where x is determined by equalizing the marginal cost of increasing x and the expected loss of exceeding the standard (100-x)% of the time. The situation is illustrated in Figure (2.15) below for the 1975 carbon monoxide standard; e can be shown to equal 2.4 gm./mi. when x is 97.5%.[47]

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[47] *Ibid.*, p. 107.

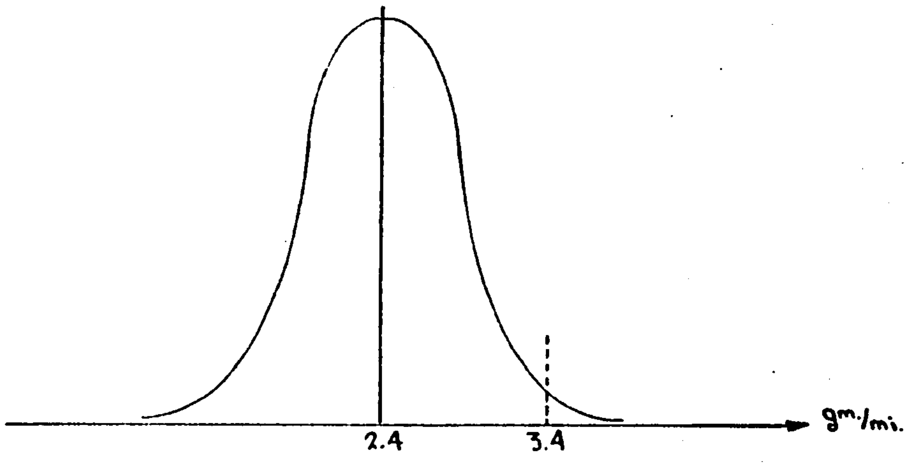

FIGURE 2.15

Notice that in this situation, quality control would be logically geared to avoid disasters; the tails of the distribution would be kept small. Were a per unit charge levied, however, this strategy would give way to one in which the entire distribution is scrutinized. The variance in emissions would then depend crucially on which emissions levels were cheaper to avoid. Since these implications are impossible to predict with the available data, we will compare price controls and quantity controls under three assumptions: the standard deviation of production induced variability in CO emissions under price regulation is 50% of, equal to, or 150% of the corresponding standard deviation under quantity regulation. If the standard deviation were reduced by 50% under prices, for instance, the variance in CO output due to production variability would fall to .12 $(gm./mi.)^2$ at the 3.4 gm./mi. level. If it were increased by 50%, on the other hand, the variance in CO emissions due to production variability would rise to 1.0 $(gm./mi.)^2$.

We have noted how production variability will create a nonzero variance in the emissions of carbon monoxide under quantity standards;

we now designate that variance σ_q^2 and assume that $\sigma_q^2 = .4\ (\text{gm./mi.})^2$ everywhere. The analysis of this chapter has emphasized the decrease in expected benefits and the increase in expected costs that such variance in output would cause. In terms of our previous notation, these losses are summarized by $(1/2)(B_{11}-C_{11})\sigma_q^2$ and are subtracted from the comparative advantage of prices that the center would compute. We have also noted a similarly created variance under price controls, indicated here by σ_p^2. This variance causes corresponding losses under prices, so that the term $(1/2)(B_{11}-C_{11})\ \sigma_p^2$ is added to the comparative advantage of prices. Finally, we have deduced profit motivated variance (σ_π^2) under prices that creates a decrease in expected benefits, an increase in expected costs, and an efficiency gain. The net of these three effects is given by $(1/2)(B_{11}+C_{11})\sigma_\pi^2$ and is also added to the center's comparative advantage. If the profit motivated variability and the production variability under price controls are independent, then the comparative advantage of prices over quantities is the sum of the above three terms taken with the indicated signs:

$$(1/2)(B_{11}+C_{11})\sigma_\pi^2 + (1/2)(B_{11}-C_{11})\sigma_p^2 - (1/2)(B_{11}-C_{11})\sigma_q^2$$

$$= (1/2)(B_{11}+C_{11})\sigma_\pi^2 + (1/2)(B_{11}-C_{11})(\sigma_p^2 - \sigma_q^2) \qquad (2.5.1)$$

Figure (2.16) reveals that at the original 1975 carbon monoxide quantity standard of 3.4 gm./mi. (96% reduction),[48] marginal benefits disappear (i.e., $B_{11} = 0$). The comparative advantage of prices then

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[48] Federal Register, 36:228, Nov., 1971, p. 22452.

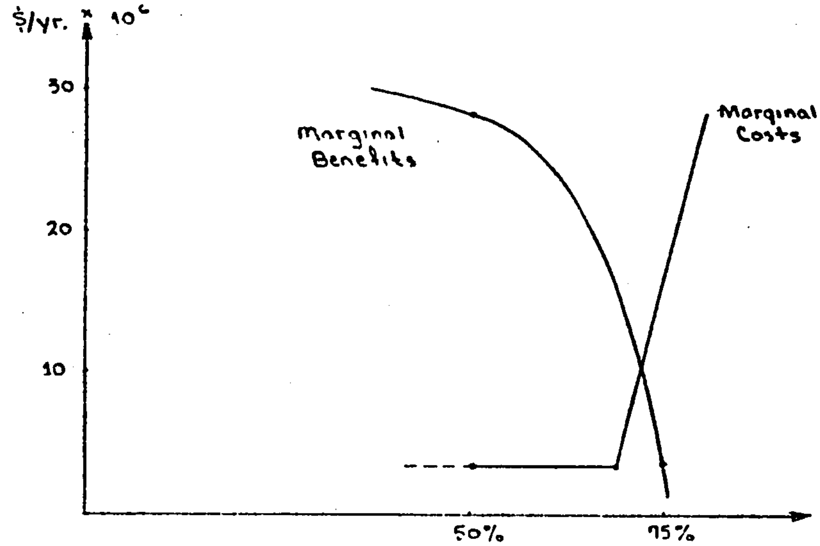

FIGURE 2.16

turns on the sign of the cost side:

$$(1/2)C_{11}(\sigma_\pi^2 + \sigma_q^2 - \sigma_p^2).$$

Table (2.4) summarizes the results for the assumed three values of σ_p^2; the values of the other parameters have been derived in the text above.

Table 2.4

Comparative Advantage of Prices at
96% Reduction of Carbon Monoxide

$B_{11} = 0$

$C_{11} = \$1.3 \times 10^7$/yr. per 1%

$\sigma_\pi^2 = 1.1$ (gm./mi.)^2

$\sigma_q^2 = .4$ (gm./mi.)^2

σ_p^2	Δ
.12	$(1/2)(1.3)(1.1 + .4 - .12) \times 10^7 > 0$
.4	$(1/2)(1.3)(1.1 + .4 - .4) \times 10^7 > 0$
1.0	$(1/2)(1.3)(1.1 + .4 - 1.0) \times 10^7 > 0$

Price controls are preferred in all three cases; a per unit charge of approximately $\$25 \times 10^{-5}$/(gm./mi.) would achieve the prescribed 96% reduction.

Consider, as a second example, a more nearly efficient requirement of 24 gm./mi. (70% reduction). The slope of the marginal benefit curve is nearly $-\$9.0 \times 10^{6}$/yr. per 1% in the region around 70%; the slope of the marginal cost curve is similarly $\$13 \times 10^{6}$/yr. per one percent. Table (2.5) summarizes the results for this case. Prices are still preferred in the first two cases; a tax of about $\$.4 \times 10^{-5}$/(gm./mi.) per automobile would regulate average emissions to the required 24 gm./mi. When the standard deviation of production variability under price controls is 50% greater than that under quantities, however, quantity standards clearly carry the day. In the center's *ex ante* view, output variation under prices is expected to be too harmful to allow, despite the inherent efficiency gains.

Table 2.5

Comparative Advantage of Prices at
70% Reduction of Carbon Monoxide

$$B_{11} = -\$9.0 \times 10^{6}\text{/yr. per } 1\%$$
$$C_{11} = \$1.3 \times 10^{7}\text{/yr. per } 1\%$$
$$\sigma_{\pi}^{2} = .3\ (\text{gm./mi.})^{2}$$
$$\sigma_{q}^{2} = .4\ (\text{gm./mi.})^{2}$$

σ_p^2	Δ
.12	$(1/2)((4.0)(.3) - (20)(.12 - .4)) \times 10^{6} > 0$
.4	$(1/2)((4.0)(.3) - (20)(0)) \times 10^{6} > 0$
1.0	$(1/2)((4.0)(.3) - (20)(1.0 - .4)) \times 10^{6} < 0$

Price controls are preferred in all three cases; a per unit charge of approximately $\$25 \times 10^{-5}$/(gm./mi.) would achieve the prescribed 96% reduction.

Consider, as a second example, a more nearly efficient requirement of 24 gm./mi. (70% reduction). The slope of the marginal benefit curve is nearly $-\$9.0 \times 10^{6}$/yr. per 1% in the region around 70%; the slope of the marginal cost curve is similarly $\$13 \times 10^{6}$/yr. per one percent. Table (2.5) summarizes the results for this case. Prices are still preferred in the first two cases; a tax of about $\$.4 \times 10^{-5}$/(gm./mi.) per automobile would regulate average emissions to the required 24 gm./mi. When the standard deviation of production variability under price controls is 50% greater than that under quantities, however, quantity standards clearly carry the day. In the center's <u>ex ante</u> view, output variation under prices is expected to be too harmful to allow, despite the inherent efficiency gains.

Table 2.5

Comparative Advantage of Prices at
70% Reduction of Carbon Monoxide

$$B_{11} = -\$9.0 \times 10^{6}\text{/yr. per } 1\%$$

$$C_{11} = \$1.3 \times 10^{7}\text{/yr. per } 1\%$$

$$\sigma_{\pi}^{2} = .3\ (\text{gm./mi.})^{2}$$

$$\sigma_{q}^{2} = .4\ (\text{gm./mi.})^{2}$$

σ_p^2	Δ
.12	$(1/2)((4.0)(.3) - (20)(.12 - .4)) \times 10^{6} > 0$
.4	$(1/2)((4.0)(.3) - (20)(0)) \times 10^{6} > 0$
1.0	$(1/2)((4.0)(.3) - (20)(1.0 - .4)) \times 10^{6} < 0$

Section 2.6: Conclusion

Either mode of control can easily allow variation in the output that is actually produced. The fundamental result of this chapter has been the vital role played by these output variations in the prices-quantities comparison. The relative magnitudes of their variances has a direct bearing, especially when the curvature of either costs or benefits approaches an extremum. Output is also varying in the context of shifting marginal costs and benefits. The simultaneity of these variations weighs on the comparison to the extent that output under either mode tends to move in the correct, or incorrect, direction. The correctness of the output shift is defined, of course, by the directions of the shifts in the marginal functions themselves.

Chapter Three

THE REGULATION OF MANY FIRMS PRODUCING ONE GOOD

The logical first extension of Chapter Two is the case in which many firms produce the same good. We therefore examine the multifirm case to determine when industry-wide price controls are preferred to industry-wide quantity controls, and when a policy mix is preferred to either of these two schemes. All of the major uncertainties introduced in the previous chapter are treated in turn, and the strength of our previous results is carefully noted.

Section 3.1: A First Model[1]

We begin our investigation of the multifirm-single product case by returning to our most elementary analytic framework. We disregard, for the moment, both the output distortion and the consumption distortion and confine uncertainty to only the random elements that influence benefits and costs. Some important results which, although they remain valid in the more complicated models, are most easily discussed and understood in the context of our simplest case.

Consider, then, a benefit function depending only on one product, denoted $B(q,\eta)$, and a total cost function

$$\sum_{i=1}^{n} C^i(q_i,\theta_i),$$

[1]This model corresponds to the case originally discussed in Section 2.1.

where C^i is the cost function facing individual firm i,[2] q_i is the output produced by firm i, and q is total output defined by

$$q = \sum_{i=1}^{n} q_i.$$

The random variables η and the θ_i once again explicitly incorporate the uncertainties over benefits and costs at the i^{th} firm, respectively, into the analysis. The center maximizes expected benefits minus costs with an accurate subjective distribution of η and the θ_i to select its optimal price and quantity orders. Each firm at the periphery reads its corresponding θ_i before selecting its profit maximizing output in response to a price order from the center. We are now equipped to investigate the comparative advantage of industry-wide price control over industry-wide quantity control in this rather simple environment.

The <u>optimal quantity order</u>, in this example, is a list of orders, $\hat{q}_{oi}$ (i = 1, ..., n), determined at the center by solving

$$\max_{(q_1,..q_2)} E(B(\sum_{i=1}^{n} q_i,\eta) - \sum_{i=1}^{n} C^i(q_i,\theta_i)).$$

The first order conditions require that the optimal orders be computed by the simultaneous solution of the following system of equations:

$$E(B_1(\sum_{i=1}^{n} \hat{q}_{oi},\eta)) = E(C_1^i(\hat{q}_{oi},\theta_i)); \; i = 1,\ldots,n. \qquad (3.1.1)$$

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[2]We assume that the cost functions all behave as the one cost function of Chapter Two. Thus,

$$C_1^i(q_i,\theta_i) > 0 \text{ and } C_{11}^i(q_i,\theta_i) > 0 \quad \text{for all } i.$$

Our assumptions concerning the shapes of the cost functions guarantee that nonnegative solutions to (3.1.1) exist. We define, for notational ease,

$$\hat{q} \equiv \sum_{i=1}^{n} \hat{q}_{oi}.$$

Having established the existence of the $\hat{q}_{oi}$, we can expand the benefit function around $\hat{q}$ and the individual cost functions around their corresponding $\hat{q}_{oi}$. We assume, as before, that the random variables enter the marginal functions only through the intercepts and that a second order approximation is sufficient; i.e., we assume that the benefit function may be approximated arbitrarily well by

$$B(q,\eta) = b(\eta) + (B' + \beta(\eta))(q - \hat{q}) + \frac{1}{2} B_{11} (q - \hat{q})^2 \qquad (3.1.2a)$$

while for the i^{th} firm, the cost function is written

$$C^i(q_i,\theta_i) = a_i(\theta_i) + (C_i' + \alpha_i(\theta_i))(q_i-\hat{q}_{oi}) + \frac{1}{2} C_{11}^i(q_i-\hat{q}_{oi})^2 \quad (3.1.2b)$$

By defining $B' = EB_1(\hat{q},\eta)$ and $C_i' = EC_1(\hat{q}_{oi},\theta_i)$, we also guarantee that

$$E\beta(\eta) = E\alpha_i(\theta_i) = 0, \ \forall i = 1,\ldots,n.^3 \qquad (3.1.3)$$

These approximations are the final assumptions required to complete the analytics of this section. Observe, before we proceed, that (3.1.1) and (3.1.3) imply immediately that $B' = C_i'$ for any i.

For any price order, p, each firm maximizes profits, given θ_i, by

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[3]The term $b(\eta)$ is defined by $B(\hat{q},\eta)$, while for any i, $a_i(\theta_i) \equiv C^i(\hat{q}_{oi},\theta_i)$.

producing the quantity, $q_i(p,\theta_i)$, that sets marginal cost equal to the given price; that is

$$p = C_i' + \alpha_i(\theta_i) + C_{11}^i(q_i(p,\theta_i) - \hat{q}_{oi}).$$

The firms' reaction functions to any price order are therefore

$$q_i = \hat{q}_{oi} - \frac{C_i' + \alpha_i(\theta_i) - p}{C_{11}^i} = h^i(p,\theta_i);$$

quite trivially, then,

$$h_1^i\,(p,\theta_i) = (C_{11}^i)^{-1}. \tag{3.1.4}$$

The center has precise knowledge of the reaction function of each firm, and selects the <u>optimal price order</u>, $\tilde{p}$, by solving

$$\max_p \{B(\sum_{i=1}^n h^i(p,\theta_i),\eta) - \sum_{i=1}^n C^i(h^i(p,\theta_i),\theta_i)\}\ .$$

The price order is thus determined by the first order condition that

$$E\ B_1(\Sigma h^i,\eta) \cdot (\Sigma h_1^i) = \sum_{i=1}^n C_1^i \cdot h_1^i\ .$$

By recalling that under price controls, the given price is always equal to the marginal costs at every firm, and by employing equation (3.1.4), we have the following implicit expression for $\tilde{p}$:

$$\begin{aligned}\tilde{p} &= E\ B_1(\sum_{i=1}^n h^i(\tilde{p},\theta_i),\eta)\\ &= B' + B_{11}\,(\sum_{i=1}^n (\frac{-C_i' + \tilde{p}}{C_{11}^i})).\end{aligned} \tag{3.1.5}$$

Since $B' = C_i'$ for any i, (3.1.5) reduces to

$$\tilde{p} = B' = C_i' \ ; \ \forall i = 1,\ldots,n. \tag{3.1.6}$$

The reaction function of the i^{th} firm to this price order is therefore

$$\tilde{q}_i(\theta_i) = \hat{q}_{oi} - \left(\frac{\alpha_i(\theta_i)}{C_{11}^i}\right). \tag{3.1.7}$$

3.1.1: The Comparative Advantage of Prices Over Quantities

We can easily compute the comparative advantage of prices in its most general form by recalling the definitions of the $\hat{q}_{oi}$ and equation (3.1.7):

$$\Delta_n = -E \int_{\sum_{i=1}^{n} \tilde{q}_i(\theta_i)}^{\sum_{i=1}^{n} \hat{q}_{oi}} B_1(q,\eta)dq + E \sum_{i=1}^{n} \int_{\tilde{q}_i(\theta_i)}^{\hat{q}_{oi}} C_1(q_i,\theta_i)dq_i$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} B_{11} \, E\left(\frac{\alpha_i(\theta_i)\alpha_j(\theta_j)}{C_{11}^i \, C_{11}^j}\right) + \frac{1}{2} \sum_{i=1}^{n} C_{11}^i \, E\left(\frac{\alpha_i(\theta_i)}{C_{11}^i}\right)$$

$$+ \sum_{i=1}^{n} E\left(\frac{\alpha_i(\theta_i)}{C_{11}^i} \beta(\eta)\right).$$

Notice that $E\left(\frac{\alpha_i}{C_{11}^i} ; \frac{\alpha_j}{C_{11}^j}\right)$ is the covariance of the output disturbances allowed by prices at firms i and j, if $i \neq j$, and is the variance of the output disturbance at one firm if $i = j$.[4] We can rewrite the expression for Δ_n as follows:

$$\Delta_n = \sum_{i=1}^{n} \frac{1}{2}(B_{11}+C_{11}^i) \text{ Var } \left(\frac{-\alpha_i(\theta_i)}{C_{11}^i}\right) + \sum_{i=1}^{n} \text{Cov } \left(\frac{-\alpha_i(\theta_i)}{C_{11}^i} ; \beta(n)\right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} B_{11} \text{ Cov } \left(\frac{-\alpha_i(\theta_i)}{C_{11}^i} ; \frac{-\alpha_j(\theta_j)}{C_{11}^j}\right). \qquad (3.1.8)$$

We will shortly be interested in the ceteris paribus effect on Δ_n of a change in the number of firms (n). To correct for the pure influence of n, we propose the following transformed cost function:

$$\Gamma^i (x_i,\theta_i) \equiv n\, C^i \left(\frac{x_i}{n},\theta_i\right) ; \forall i = 1,\ldots,n,$$

where x_i is total output.[5] Observe that Γ^i may be interpreted as total cost given as a function of total output under the assumption that each firm is an exact duplicate of the i^{th}. It will be Γ^i and not C^i that will be held constant to insure that we are capturing only the effects of a change in n and not the effects of some secondary production/cost changes.

The characteristics that make the above transformation particularly useful are summarized below:

$$\Gamma_1^i = C_1^i; \ i = 1,\ldots,n;$$
$$\Gamma_{12}^i = C_{12}^i; \ i = 1,\ldots,n; \text{ and}$$
$$\Gamma_{11}^i = C_{11}^i; \ i = 1,\ldots,n.$$

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[4] We see this by observing that $E\alpha_i(\theta_i)\cdot\alpha_j(\theta_j)=E(\alpha_i(\theta_i)-E\alpha_i)(\alpha_j(\theta_j)-E\alpha_j)$.

[5] Weitzman, op. cit., p. 23.

We have, as a direct consequence of these relations that the variance-covariance matrix of marginal cost, and thus output disturbances under prices, has survived the transformation intact. Equation (3.1.8) can now be rewritten

$$\Delta_n = (1/n)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} B_{11}\, E(\alpha_i \alpha_j / \Gamma^i_{11} \Gamma^j_{11})$$

$$+ (1/n) \sum_{i=1}^{n} (C^i_{11} \text{ Var } (\alpha_i/\Gamma^i_{11}) + \text{Cov } (-\alpha_i/\Gamma^i_{11}\ ;\beta) \qquad (3.1.9)$$

$$= (1/n)^2 \sum_{i=1}^{n} (B_{11} + \Gamma^i_{11}) \text{ Var } (\alpha_i/\Gamma^i_{11})$$

$$+ (1/n)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} B_{11} \text{ Cov } (\alpha_i/\Gamma^i_{11}\ ;\ \alpha_j/\Gamma^j_{11})$$

$$+ (1/n) \sum_{i=1}^{n} \text{Cov } (-\alpha_i/\Gamma^i_{11}\ ;\beta). \qquad (3.1.10)$$

Even casual consideration of (3.1.9) reveals that, in this initial multifirm model, the spirit of the single firm case has been preserved. We see that an increase in the absolute magnitude of B_{11} causes the negative impact of total output variation to increase; Δ_n is diminished. An increase in C^i_{11}, and thus in Γ^i_{11} for a given n, similarly reduces output variation under prices and favors price control. The second term in (3.1.10) registers the influence of output covariance across firms; it can best be explained by comparing the case in which these covariances are all zero with a case in which only one covariance is nonzero. Suppose that covariance is between firms 1 and 2. If $\text{Cov}(\alpha_1/\Gamma^1_{11};\alpha_2/\Gamma^2_{11}) > 0$, outputs of firms 1 and 2 tend to vary in the same direction under prices and therefore increase total output variation, over the case of complete interfirm independence, by amplifying each other. Such increased variation is detrimental and is an additional bias against price control ($B_{11}\mathrm{Cov}(\frac{\alpha_1}{\Gamma^1_{11}};\frac{\alpha_2}{\Gamma^2_{11}}) < 0$). Conversely, when $\mathrm{Cov}(\frac{\alpha_1}{\Gamma^1_{11}};\frac{\alpha_2}{\Gamma^2_{11}}) < 0$, outputs of 1 and 2 tend to vary in opposite directions and cancel; the resulting decrease in the variation of total output is a positive bias for prices over the independent case and $B_{11}\mathrm{Cov}(\frac{\alpha_1}{\Gamma^1_{11}};\frac{\alpha_2}{\Gamma^2_{11}}) > 0$.

3.1.2: The Effect of a Change in the Number of Firms

We now consider the case in which all of the firms are identical, so that we can totally isolate the effect of a change in the number of firms, n, on the comparative advantage of prices over quantities, i.e.,

$$\Gamma^i_{11} \equiv \Gamma_{11};\ i = 1,\ldots,n;\ \text{and}$$

$$\mathrm{Var}(\alpha_i/\Gamma^i_{11}) \equiv \sigma^2;\ i = 1,\ldots,n.$$[6]

We further assume that the θ_i are distributed such that the correlation between the marginal costs of any two firms is equal to one particular constant:

$$\mathrm{Cov}(\frac{\alpha_i}{\Gamma^i_{11}};\frac{\alpha_j}{\Gamma^j_{11}}) = \rho\sigma^2;\ i \neq j.$$

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[6]These two statements are not precisely equivalent to requiring that the firms be identical, but Γ_{11} and σ^2 are the two parameters that appear in our expression for the comparative advantage of prices.

The parameter ρ is the constant correlation coefficient.[7] The comparative advantage of prices is now

$$\Delta_n = (\frac{1}{n^2}) B_{11} [\frac{n}{2} \sigma^2 + n(n-1) \frac{\rho\sigma^2}{2}] + \frac{1}{2} \Gamma_{11} \sigma^2 + Cov(\frac{\alpha}{\Gamma_{11}} ; \beta)$$

$$= \rho(B_{11} \frac{\sigma^2}{2} + \Gamma_{11} \frac{\sigma^2}{2}) + (1-\rho)(\frac{1}{n} B_{11} \frac{\sigma^2}{2} + \Gamma_{11} \frac{\sigma^2}{2})$$

$$+ Cov(\frac{\alpha}{\Gamma_{11}} ; \beta). \qquad (3.1.11)$$

In considering a <u>ceteris paribus</u> change in n, we will be holding the benefit function, the transformed cost function, σ^2, and ρ fixed.

Observe that with any finite degree of interfirm independence, a <u>ceteris paribus</u> increase in the number of firms contributing to the total output of the single good in question produces a correspondingly greater comparative advantage of prices. A few remarks are in order before the economic forces behind this result are explored. Notice, first of all, that a large n does not guarantee that Δ_n will be positive.[8] Furthermore, this result in no way depends on our assumption that all firms are identical; consideration of (3.1.10) directly would reach the same conclusion by means of more arduous reasoning. The identical firm assumption, therefore, serves only to render the exposition simpler.

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[7]The parameter ρ is constrained by [-1, 1], and although it is usually positive in practice, it will not be so assumed.

[8]The crucial limit is

$$\lim_{n \to \infty} \Delta_n = (\frac{B_{11}}{2}(\frac{\sigma}{\Gamma_{11}})^2) + (\frac{\Gamma_{11}}{2}(\frac{\sigma}{\Gamma_{11}})^2)$$

which need not be positive.

Turning to the economic genesis of the result, the first effect that comes to mind is the <u>ex post</u> efficiency gain afforded price controls by their guarantee that marginal costs are equal across firms; that is

$$C_1^i\ (\tilde{q}_i(\theta_i),\theta_i) = C_1^j\ (\tilde{q}_j(\theta_j),\theta_j) = \tilde{p} \quad \forall i;j=1,\ldots,n.$$

Price controls, therefore, automatically screen high cost producers and encourage them to produce less. Low cost producers are similarly encouraged to produce more. Under quantities, however, we have that

$$C_1^i\ (\hat{q}_{oi},\theta_i) \neq C_1^j\ (\hat{q}_{oj},\theta_j) \quad \forall i;j = 1,\ldots,n,$$

except on a set of measure zero. The very special case in which the θ_i are perfectly correlated ($\rho = 1$) lies within that set. The efficiency gain disappears, in that case, because marginal costs are equal across all firms even when quantity controls are imposed.[9] Our observation requires, therefore, at least a finite degree of interfirm independence.

A second influence can be uncovered if we compute the set of points ($\bar{q}_{oi}$: $i = 1,\ldots,n$) such that

$$\sum_{i=1}^{n} \hat{q}_{oi} = \sum_{i=1}^{n} \bar{q}_{oi} \ ;$$

$$C_1^i\ (\bar{q}_{oi},\theta_i) = C_1^j\ (\bar{q}_{oj},\theta_j) \ ; \quad \forall i;j = 1,\ \ldots,\ n \ ;$$

that is, the $\bar{q}_{oi}$ are selected not only to equalize marginal costs across firms, but also to maintain total output at the level prescribed by

[9]For identical firms, $\hat{q}_{oi} = \hat{q}_{oj}$.

quantity controls. We can solve for the $\bar{q}_{oi}$ in the general case using Cramer's Rule and induction on the number of firms (see Appendix B):

$$\bar{q}_{oi} = \hat{q}_{oi} - \frac{(n-1)\alpha_i(\theta_i) - \sum_{\substack{k=1 \\ k\neq i}}^{n} \alpha_k(\theta_k)}{n\, C_{11}^{i}} .$$

We have assumed, in this subsection, that all of the firms are identical so that

$$\bar{q}_{oi} = \hat{q}_{oi} - \frac{(n-1)\alpha(\theta_i) - \sum_{\substack{k=1 \\ k\neq i}}^{n} \alpha(\theta_k)}{n\, C_{11}} . \qquad (3.1.12)$$

Equation (3.1.12) will illustrate the second effect: a diversification gain afforded price controls as the number of firms increases.

Observe, first of all, that if the θ_i are perfectly correlated, then $\alpha(\theta_i) = \alpha(\theta_j)$ for all i and j and $\hat{q}_{oi} = \bar{q}_{oi}$. In this case, there can be no gain in setting marginal costs equal because they already are equal under quantities; we have noted this point before. On the other extreme, if the θ_i are independently distributed, then

$$\sum_{\substack{k=1 \\ k\neq i}}^{n} \alpha(\theta_k) = (n-1)\, E\,(\alpha(\theta)) = 0$$

as n grows, and

$$\lim_{n\to\infty} \bar{q}_{oi} = \lim_{n\to\infty} \hat{q}_{oi} - \frac{(n-1)\alpha(\theta_i)}{n\, C_{11}}$$

$$= \hat{q}_{oi} - \frac{\alpha(\theta_i)}{C_{11}} \equiv \tilde{q}_i(\theta_i).$$

In the limit, therefore, we can set an output for each firm so that the marginal cost of each firm is precisely equal to the optimal price order without moving total output from the level set under optimal quantity control. Most cases lie between the two extremes, but the intuition these extremes provide can certainly be extrapolated. The gain from diversification lies in the ability, as n increases, to set marginal costs of each firm closer to $\tilde{p}$ without altering total output. Put another way, the total output variation that results from setting marginal costs at each firm equal to $\tilde{p}$ becomes monotonically smaller as the number of firms increases. Since it is that output variation that "hurts" prices, the diminishing of that variation is a positive bias in the comparative advantage of prices.

The following proposition serves to summarize the import of this subsection.

Proposition 1:

> A ceteris paribus increase in the number of firms contributing to the total output creates a corresponding increase in the comparative advantage of industry-wide price control over industry-wide quantity control, given any finite degree of interfirm independence.

We have seen that there are two economic forces that simultaneously contribute to the genesis of this proposition. The first is the efficiency gain caused by the equalization of marginal costs across firms under price control. The second is a diversification gain that diminishes the variation of total output under the same industry-wide price specification.

Section 3.2: Mixing Policies Within an Industry[10]

In this section, we begin to investigate the existence of circumstances in which a mixed policy set--i.e., the regulation of some of the firms in the industry by prices; the rest, by quantities--would be preferable to uniform industry-wide control by either mode. We still conform to the model specified at the beginning of Section 3.1; in summary:

(1) The center maximizes expected benefits minus expected costs with the correct distribution for the random variables.

(2) The peripheral firms maximize profits by observing the value of θ_i that appears and setting the ordered price equal to marginal costs.

(3) There is no output distortion under quantities and no consumption distortion at all.

Consider, as a first example, the case in which one good is produced by n firms that are divided into two groups by differences in their cost functions. Assume, to be more precise, that we can number the firms so that the first m confront cost functions of the form

$$C^1(q_i,\theta_i) = a_1(\theta_i) + (C_1' + \alpha_1(\theta_i))(q_i-\hat{q}_{o1}) + \frac{1}{2} C_{11}^1(q_i-\hat{q}_{o1}),$$

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[10] Until we consider the general case in Section 3.2.3, we assume for expositional ease that the θ_i and η are independent.

while the remaining (n-m) face

$$C^2(q_i,\theta_i) = a_2(\theta_i) + (C_2' + \alpha_2(\theta_i))(q_i-\bar{q}_{o2}) + \frac{1}{2} C_{11}^2(q_i-\bar{q}_{o2})^2.$$

We further assume that the correlation coefficients of output variation under prices are unequal, both within and between the two groups. This final assumption is mathematically specified as follows:[11]

$$\mathrm{Var}(\alpha_i/\Gamma_{11}^i) = \sigma^2 \; ; \; i = 1,\ldots,m;$$

$$\mathrm{Cov}(\alpha_i/\Gamma_{11}^i;\alpha_j/\Gamma_{11}^j) = \rho_1\sigma^2; \; i \text{ and } j = 1,\ldots,m \text{ with } i \neq j;$$

$$\mathrm{Var}(\alpha_k/\Gamma_{11}^k) = t\sigma^2; \; k = (m+1), \ldots, n;$$

$$\mathrm{Cov}(\alpha_k/\Gamma_{11}^k;\alpha_\ell/\Gamma_{11}^\ell) = t\rho_2\sigma^2; \; k \text{ and } \ell = (m+1),\ldots,n \text{ with } \ell \neq k; \text{ and}$$

$$\mathrm{Cov}(\alpha_i/\Gamma_{11}^i;\alpha_k/\Gamma_{11}^k) = \sqrt{t}\,\rho_3\sigma^2 \; ; \; i = 1,\ldots,m \text{ and } k = (m+1),\ldots,n.$$

The comparative advantage of prices for this example can be computed directly from (3.1.10):

$$\begin{aligned}\Delta_n' = \frac{m}{n} \{&\rho_1(\frac{m}{n} B_{11} \frac{\sigma^2}{2} + \Gamma_{11} \frac{\sigma^2}{2}) + (1-\rho_1)(\frac{1}{n} B_{11} \frac{\sigma^2}{2} + \Gamma_{11} \frac{\sigma^2}{2})\} \\ &+ (\frac{n-m}{n})\{\rho_2(\frac{n-m}{n} B_{11} \frac{t\sigma^2}{2} + \frac{\Gamma_{11}t\sigma^2}{2}) + (1-\rho_2)(\frac{B_{11}t\sigma^2}{2n} + \frac{\Gamma_{11}t\sigma^2}{2})\} \\ &+ (\frac{m}{n})(\frac{n-m}{n})(B_{11}\rho_3\sigma^2\sqrt{t}).\end{aligned} \quad (3.2.1)$$

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[11] We will shortly correct for the pure effect of n, so that we list these assumptions in their transformed notation. These conditions are simultaneously constraints on the variance-covariance matrix of marginal costs. The covariance between the groups appears in its peculiar form because given any two random variables, x and y, the correlation coefficient is defined

$$\rho_{xy} \equiv (\sigma_{xy}^2/\sqrt{\sigma_x^2\,\sigma_y^2}).$$

When the two groups of firms are independent ($\rho_3 = 0$), Δ'_n equals the convex sum of a modified expression for the comparative advantage of prices for each group taken alone. The modification, exhibited by the fractional coefficients in (3.2.1), reflects the diversification gain afforded price control by the grouping; each group, taken individually, can now affect only a *fraction* of the total industry-wide output. Each group can therefore provide only a fraction of the variance in total output that appears in the benefit function. Within each bracket of (3.2.1), then, is the comparative advantage of prices over quantities for each group, taken in the context of that group's *relative position in the industry*. We make that distinction notationally by rewriting (3.2.1) as follows:

$$\Delta'_n = (m/n)\ \Delta^1\ (m/n) + \left(\frac{n-m}{n}\right)\ \Delta^2\ \left(\frac{n-m}{n}\right) + \left(\frac{m}{n}\right)\left(\frac{n-m}{n}\right)\ B_{11}\rho_3\sigma^2\sqrt{t}. \tag{3.2.1)'}$$

Postponing a detailed discussion of the origins of this change until later in the section, we can easily note an example in which a policy mix would be preferred. Suppose that, taking its position in the industry into consideration, prices are preferred in group one, while quantities are similarly favored in group two. In our new notation,

$$\Delta^1(m/n) > 0 \quad \& \quad \left(\frac{n-m}{n}\right) < 0.$$

A simple counting argument reveals that a mix with price controls over group 1 and quantity controls over group 2 outranks industry-wide control of either kind when $\rho_3 = 0$.[12]

3.2.1: Optimal Decisions Under a Mixed Scheme

Having demonstrated the existence of cases in which policy mixes are preferable, we will shortly pursue generality in several directions. It is necessary, however, to demonstrate that the quantity decisions made under a mixed scheme either at the center, or at the periphery in response to a price order from the center, are the same as they would have been under a uniform control. We consider, to that end, a potential mix in which the first m firms face price controls and maximize profits, while the last (n-m) firms operate under direct quantity regulation. The center seeks to maximize expected benefits minus costs with respect to its control variables p' and (q_i': $i = (m+1), \ldots, n$), having full <u>ex ante</u> knowledge of the price response curves of the first m firms:[13]

$$q_i'(p',\theta_i) \equiv h^i(p',\theta_i)$$

The center therefore solves

$$\max_{(p';\, q'_{m+1},\ldots,q'_n)} \{E(B((\sum_{i=1}^{m} h^i(p',\theta_i) + \sum_{j=m+1}^{n} q_j'),n) - \sum_{i=1}^{m} C_i(h^i,\theta_i) - \sum_{j=m+1}^{n} C_j(q_j',\theta_j))\}$$

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[12] We are essentially maximizing expected benefits minus costs over four alternatives that include the two previously available choices. We can therefore do no worse; this example is constructed so that we do better.

[13] Since firms are profit maximizers, they are concerned only with the price at which they can sell their product and their own marginal cost schedule, <u>not</u> the type of control that other firms are facing. Their price response curves are therefore the same as those introduced in Section 3.1.

for the optimal control mix, designated $\{\tilde{p}'; \hat{q}'_{m+1}, \ldots, \hat{q}'_n\}$.

Expanding the benefit and cost functions around $\hat{q}$ and the $\hat{q}_{oi}$ as defined in section 3.1, we see immediately from the last (n-m) first order conditions that

$$\hat{q}'_{oi} = \hat{q}_{oi} \text{ for any } i = (m+1), \ldots, n.$$

The remaining first order condition requires, in addition, that

$$\tilde{p} = \left\{\frac{E(B_1 \cdot \sum_{i=1}^{m} h_1^i)}{E \sum_{i=1}^{m} h_1^i}\right\}$$

$$= E(B_1((\sum_{i=1}^{m} h_i(\tilde{p},\theta_i) + \sum_{i=m+1}^{n} \hat{q}'_{oi}),n)$$

$$= B' + B_{11}(\sum_{i=1}^{m} (\frac{-C_i' + \tilde{p}}{C_{11}^i})).$$

We recall that $B' = C_i'$ for all i and conclude that $p = B' = C_i'$.[14] Since both the optimal price order and the price response curves for every firm have remained the same, the quantity response to that price order, denoted $\tilde{q}_i'(\theta_i)$, is similarly identical:

$$\tilde{q}_i'(\theta_i) = \hat{q}_{oi} - \frac{\alpha_i(\theta_i)}{C_{11}^i}, \quad \forall i = 1, \ldots, m.$$

Observe finally that the preceding argument is independent of the ordering placed on the firms as well as the number of firms chosen to be

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[14]See the argument outlined in Section 3.1 for a more complete demonstration.

regulated by price control. We have demonstrated, therefore, that the quantity decisions made at either seat of authority under a mixed policy are precisely those that would have been made under the corresponding industry-wide control.

3.2.2: Policy Mixes with Positive Intergroup Correlation

We now return to the assumption that there are two types of firms, distinguished by their respective cost functions. We further require, for the moment, that the random variables influencing the various cost functions be positively correlated across groups (i.e., $\rho_3 > 0$). Consider a case in which quantity controls are preferred when the industry is placed under uniform regulation (i.e., $\Delta_n < 0$). We propose, in contrast to industry-wide quantity control, a policy mix that regulates group 1 by prices and group 2 by quantities. The comparative advantage of that mix over uniform quantity control is given by

$$\Delta_n(pq/qq) = \frac{1}{2} m\, C_{11}^{1}\, \sigma^2 + \frac{1}{2} m\, B_{11}\, \sigma^2 + \frac{1}{2} m\, (m-1)\rho_1 \sigma^2 \qquad (3.2.2)$$

Correcting for the pure effect of the total number of firms,

$$\begin{aligned}\Delta_n(pq/qq) &= (m/n)[(\rho_1/2)[(\frac{mB_{11}}{n})(\sigma/\Gamma_{11}^{1})^2 + \Gamma_{11}^{1}(\sigma/\Gamma_{11}^{1})^2] \\ &\quad + (\frac{1-\rho_1}{2})[(\frac{B_{11}}{n})(\sigma/\Gamma_{11}^{1})^2 + \Gamma_{11}^{1}(\sigma/\Gamma_{11}^{1})^2]] \\ &= \Delta^{1}(m/n) \qquad (3.2.2)'\end{aligned}$$

The comparative advantage of this mix over quantities is precisely the modified comparative advantage of prices over quantities in group 1 when the "position" of group 1 in the industry is taken into account. That position is indicated by the preliminary coefficient (m/n).

Were we to reverse the mix, we would similarly find the comparative advantage of that second mix over uniform quantity control to be

$$\Delta_n(qp/qq) \quad = \quad \Delta^2(\frac{n-m}{n}).$$

We can rank these two mixes by comparing the modified comparative advantages of prices in the two groups taken individually; if either, or both,[15] of these statistics is positive, the group having the higher comparative advantage should be switched to price control when the mix is imposed.

It is also useful to contrast these two mixes with industry-wide price control in the case in which industry-wide price controls are favored. Recall that the first mix regulates group 1 by prices and group 2 by quantities. Correcting as before for the pure effect of n, we can see that the comparative advantage of mix 1 over uniform price control is given by

$$\begin{aligned}\Delta_n(pq/pp) = & -(\frac{n-m}{n})[(\rho_2/2)[(\frac{n-m}{n})\ B_{11}\ t\ (\sigma/r^2_{11})^2 + tr^2_{11}\ (\sigma/r^2_{11})^2] \\ & + (\frac{1-\rho_2}{2})[(\frac{B_{11}t}{n})(\sigma/r^2_{11})^2 + tr^2_{11}\ (\sigma/r^2_{11})^2]] \\ & - B_{11}\ (\frac{m}{n})(\frac{n-m}{n})\ \rho_3\ \sqrt{t}\ (\sigma^2/r^1_{11}r^2_{11}) \\ = & -\Delta^2\ (\frac{n-m}{n}) - B_{11}\ (\frac{m}{n})(\frac{n-m}{n})\ \rho_3\ \sqrt{t}\ (\sigma^2/r^1_{11}r^2_{11}). \qquad (3.2.4)\end{aligned}$$

The first term of (3.2.4) is totally expected in light of the previous example; it is the comparative advantage of quantities over prices in

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[15]Since $\rho_3 > 0$, it is possible for $\Delta^1(m/n) > 0$ and $\Delta^2(n-m/n) > 0$ and still have $\Delta_n < 0$ (see equation (3.2.1)).

group 2, still taking that group's relative position in the industry into account. The second term should be no surprise, either. When the entire industry is under price control, the positive correlation of costs, and thus output variation, between the two groups tends to magnify the effect of individual output variation on the variance of total output. This tendency produces an additional negative bias against price, over the case of group independence, registered by

$$B_{11}\left(\frac{m}{n}\right)\left(\frac{n-m}{n}\right) \rho_3 \sqrt{t}\, (\sigma^2/\Gamma^1_{11}\Gamma^2_{11}) < 0. \qquad (3.2.4a)$$

When group 1 is regulated by quantities, however, the output of each firm in that group is fixed (in the current simple model), and the effective covariance across groups is zero. The mix in question, therefore, collects a positive bias over a regime of uniform price control equal to the absolute magnitude of (3.2.4a).

When the mix is reversed, a second comparative advantage can be computed, and similar manipulation reveals that

$$\Delta(qp/pp) = -\Delta^1\left(\frac{m}{n}\right) - \left(\frac{m}{n}\right)\left(\frac{n-m}{n}\right) B_{11}\, \rho_3 \sqrt{t}\, (\sigma^2/\Gamma^1_{11}\Gamma^2_{11}).$$

We can once again rank the two mixes by comparing the values assumed by the two modified comparative advantages of prices; this time, however, the group having the (algebraically) larger comparative advantage should remain under price control when the mix is enacted.

The following proposition summarizes our results thus far:

<u>Proposition 2:</u>

Suppose that $\rho_3 > 0$. If price (quantity) regulation is preferred in either group 1 or group 2, taking into account the relative position of these groups in the total industry, even though quantity (price) control is preferred across the industry taken as a whole, then a mix that controls that group with prices (quantities) and the remainder of the industry with quantities (prices) ranks higher than uniform quantity (price) regulation.

To be sure, this is a result of rather limited scope. It will nonetheless serve two purposes in subsequent subsections. First of all, it will be applied to an even more specific example in which we will clarify the often quoted, but thus far poorly rationalized phrase, "taking into account its position in the industry." It appears, in addition, as an intuitive precursor of a similar result cast in a more general forum. We will, however, complete our examination of this two-group model by discussing the effects of negative intergroup correlation before proceeding in those directions.

The algebra of the previous results survives the change in the sign of ρ_3 completely intact. The interpretations of the various cases are, however, altered drastically. For instance, when the base for comparison is industry-wide quantity control, we cannot simultaneously observe positive values for both $\Delta^1(m/n)$ and $\Delta^2(n-m/n)$. The choice between mixes is therefore simple to make: choose to put under price control that group whose modified comparative advantage of prices is positive. The complement of that group will, of course, remain under quantity regulation.

Even greater changes are created when the basis for comparison is industry-wide price control. The second term of (3.2.4) is then positive and represents a gain over intergroup independence resulting from the dampening effect on the variance of total output of the negative across-group correlation in marginal costs. In order that a mix be preferred, in this case, it must therefore overcome this positive bias toward uniform price regulation; there must exist a group which, taken alone, strongly favors quantity control.

3.2.3: An Illustrative Digression

We return briefly, in this subsection, to the world in which there are n identical firms producing the one product. Each firm faces the cost function:

$$C(q_i,\theta_i) = a(\theta_i) + (C' + \alpha(\theta_i))(q_i-\hat{q}_o) + \frac{1}{2}C_{11}\,(q_i-\hat{q}_o)^2,$$

while the θ_i are distributed such that

$$Var(\alpha(\theta_i)) = \sigma^2;\ i = 1, \ldots, n, \text{ and}$$

$$Cov(\alpha(\theta_i);\alpha(\theta_j)) = \rho\sigma^2;\ i \text{ and } j = 1, \ldots, n;\ i \neq j.$$

The meaning of the phrase "taking into account a firm's position in the industry" can be demonstrated, under these conditions, by the following corollary to Proposition 2:

<u>Corollary 1</u>:

Suppose that quantity controls are preferred when the industry is taken as a whole. There exists a subset of $\bar{m}$ firms such that a mix that controls those $\bar{m}$ firms by prices and the remaining $(n-\bar{m})$ firms by quantities is favored to industry-wide quantity control if and only if

$$(1/n) \left| \frac{B_{11}}{C_{11}} \right| \leq 1. \qquad (3.2.5)$$

The corollary follows directly from manipulation of equation (3.2.2) in the context of condition (3.2.5); a formal proof is recorded in Appendix C. Equation (3.2.5) is then seen to guarantee the superiority of the mix of one firm under price control and (n-1) others under quantities over a uniform quantity regulation of the entire industry. The intuition of the phrase can now be constructed more completely by a thorough study of the genesis of that equation.

We wish to compare the one firm/one product case to the current model in which the firm in question is essentially $(1/n)^{th}$ of the industry. Recall that in the one firm case, the comparative advantage of prices over quantities is[16]

$$\Delta = \left(\frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right)$$

The first term registers the efficiency gain that is derived by insuring that the firm is setting marginal costs equal to the prescribed price, diminished by the loss created by extra costs due to output variation. $(B_{11}\,\sigma^2/2\Gamma_{11}^2)$ is similarly the loss due to output variation that is registered in the benefit function. When the firm is embedded in an

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[16]The precise analog of the one firm case in this section is a set of n perfectly correlated duplicates of one firm. Then,

$$\Delta = \frac{n}{2}\, C_{11}\left(\frac{\sigma}{C_{11}}\right)^2 + \frac{n^2}{2}\, B_{11}\left(\frac{\sigma}{C_{11}}\right)^2$$

Correcting for the pure effect of n as before,

$$\Delta = \frac{1}{2}\,\Gamma_{11}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{1}{2}\, B_{11}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 = \frac{1}{2}\,(B_{11}+\Gamma_{11})\mathrm{Var}(\alpha/\Gamma_{11}).$$

industry of (n-1) identical other firms that are controlled directly by quantities, however, the correspondingly diminished efficiency gain is given by

$$E\{(n-1)\int_{\hat{q}_o}^{\hat{q}_o} C_1(q_i,\theta_i)dq_i + \int_{\hat{q}_o-\frac{\alpha(\theta_i)}{C_{11}}}^{\hat{q}_o} C_1(q_i,\theta_i)dq_i\} = 0 + \frac{C_{11}}{2}(\frac{\sigma}{C_{11}})^2 = \frac{\Gamma_{11}}{2n}(\frac{\sigma}{\Gamma_{11}})^2 \tag{3.2.6}$$

Observe that the gain is reduced by a factor of (1/n) from the single firm case. We can see, in like manner, that the variation induced loss in benefits is now

$$E\{\int_{n\hat{q}_o-\frac{\alpha(\theta_i)}{C_{11}}}^{n\hat{q}_o} B_1(q,n)dq\} = \frac{B_{11}}{2}(\frac{\sigma}{C_{11}})^2 = \frac{B_{11}}{2n^2}(\frac{\sigma}{\Gamma_{11}})^2 \tag{3.2.7}$$

i.e., it is diminished, in absolute terms, by $(1/n)^2$. As we noted above, when the single firm is incorporated into a large industry, it can influence only a fraction of the total output of that industry. Equations (3.2.6) and (3.2.7) reveal the asymmetry of this diminishing effect on the two sides of the loss function. It is therefore possible that

$$|B_{11}\sigma^2/2\Gamma_{11}^2| > (\Gamma_{11}\sigma^2/2\Gamma_{11}^2) > \frac{1}{n}|B_{11}\sigma^2/2\Gamma_{11}^2| ,$$

in which case the firm taken within the industry predicts that prices are preferred, while that same firm taken as the entire industry predicts the exact opposite:

$$\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma}\right)^2 < 0 \quad \text{and}$$

$$\left(\frac{1}{n}\right)\left(\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 > 0.$$

When we write of a firm's position in the industry, it is precisely this asymmetrical reduction in the influence of individual output variation on total output variation to which we refer.

Several remarks should be made, for the sake of completeness, before we pass to a more general example. The converse of Corollary 1 is false whenever $\rho > 0$; were prices preferred industry-wide, there would exist no single firms for which quantity controls would be preferred, even when they are taken alone in the context of their place in the industry. To be more precise,

$$\Delta_n = \rho\left(\frac{B_{11}}{2}(\sigma/\Gamma_{11})^2 + \frac{\Gamma_{11}}{2}(\sigma/\Gamma_{11})^2\right) + (1-\rho)\left(\frac{B_{11}}{2n}(\sigma/\Gamma_{11})^2 + \frac{\Gamma_{11}}{2}(\sigma/\Gamma_{11})^2\right) > 0$$

is sufficient to guarantee that[17]

$$\Delta(1/n) = \frac{1}{n}\left(\frac{1}{n}\left(\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) + \left(\frac{\Gamma_{11}}{2}\right)\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) > 0.$$

We are therefore dealing with a case in which the efficiency gains of price controls dwarf the losses attributed to total output variation, even when that variation is amplified across the industry by positively

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[17]That conclusion follows from the following sequence of inequalities:

$$0 < \Delta_n \leq \rho\left(\frac{1}{n}\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) + (1-\rho)\left(\frac{1}{n}\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right)$$

$$= \left(\frac{1}{n}\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right)$$

$$= \Delta(1/n)$$

correlated marginal costs.

We can also demonstrate that the assumption that ρ be positive is crucial to the validity of the corollary. To do so, we assume that $\rho < 0$, recall that the industry-wide preference of quantities requires that $\Delta_n < 0$, and look for a contradiction. If the mix of one firm under prices and the remaining (n-1) firms under quantities is preferred to uniform quantity regulation of the entire industry, then

$$\Delta(1/n) = (1/n)\left(\frac{B_{11}}{2n}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) > 0.$$

Since

$$(1/n)\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 \geq \frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2$$

and $\rho < 0$, we have immediately that

$$\rho\left[(1/n)\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right] \leq \rho\left[\frac{B_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{11}{2})^2\right]$$

so that

$$\Delta_n \geq \rho\left(\frac{B_{11}}{2n}\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) + (1-\rho)\left(\left(\frac{B_{11}}{2n}\right)\left(\frac{\sigma}{\Gamma_{11}}\right)^2 + \frac{\Gamma_{11}}{2}\left(\frac{\sigma}{\Gamma_{11}}\right)^2\right) > 0.$$

We therefore have our contradiction. If $\rho < 0$ and uniform quantity control is preferred to uniform price control, there cannot exist firms for which price controls would be preferred when they are viewed individually within the industry.

When $\rho < 0$, however, all is not lost. If prices are preferred over the entire industry, it is necessary, but <u>not</u> sufficient, that there exist a firm for which quantities are favored, in the context of its place in the industry, for the mix of that firm under quantity control and (n-1) other firms under price control to be favored to uniform price control across the industry. The asymmetry in results occurs because the mix must overcome the resulting loss in diversification that is afforded prices in this case by the dampening effect on total output variation of a negative correlation in costs across firms. When equation (3.2.4) is considered in this example, the second term is a negative bias against the mix, since B_{11} and ρ are both negative.

3.2.4: General Conditions for a Mix

We finally return to the general case in which there are simply n firms producing one product; no assumptions will be made about their similarity, except that they all face cost functions of the standard form:

$$C^i(q_i,\theta_i) = a_i(\theta_i) + (C_i' + \alpha_i(\theta_i))(q_i-\hat{q}_{oi}) + \frac{1}{2} C_{11}^i (q_i-\hat{q}_{oi})^2$$

The benefit function remains intact and we retain the definition

$$B(q,\eta) = b(\eta) + (\beta(\eta)+ B')(q-\hat{q}) + \frac{1}{2} B_{11} (q-\hat{q})^2$$

from Section 3.1. If there exist independent subgroups of firms, they may be dealt with individually, as long as we keep in mind their position in the industry, thereby recognizing the fractional impact of their cumulative product on the variation of total industry output. We can therefore assume, without loss of generality, that there does not exist a subset of firms that is totally independent of the rest of the industry. Notice, however, that this assumption in no way precludes a zero correlation between any two firms.

To capture maximum generality, we begin with an arbitrary mix in which the first m firms are regulated by quantity controls and the remaining (n-m) are regulated by prices. The index m is allowed to range between zero and n. Calling the above mix 1, we will now contrast a second mix which duplicates the first but for the m^{th} firm; that firm is switched to price control. The comparative advantage of mix 2 over mix 1 is then simply:[18]

$$(\text{mix } 2/\text{mix } 1) = E\int_{\hat{q}_{om} - \frac{\alpha_m}{C_{11}^m}}^{\hat{q}_{om}} C_1^m(q_m,\theta_m)dq_m - E\int_{\hat{q} - \sum_{i=m}^{n}\alpha_i/C_{11}^i}^{\hat{q}_o - \sum_{i=m+1}^{n}\alpha_i/C_{11}^i} B_1(q,\eta)dq$$

$$= \frac{1}{2} C_{11}^m \text{ Var}(\alpha_m/C_{11}^m) + \text{Cov}(\alpha_m/C_{11}^m;\beta)$$

$$+ B_{11}[\sum_{i=m+1}^{n} \text{Cov}(\alpha_m/C_{11}^m;\alpha_i/C_{11}^i) + \frac{1}{2}\text{ Var}(\alpha_m/C_{11}^m)]$$

$$= \frac{1}{n}[\frac{1}{2}\Gamma_{11}^m \text{ Var}(\alpha_m/\Gamma_{11}^m) + B_{11}(\sum_{i=m+1}^{n} \text{Cov}(\alpha_m/\Gamma_{11}^m;\alpha_i/\Gamma_{11}^i)$$

$$+ \text{Cov}(\alpha_m/\Gamma_{11}^m;\beta) + \frac{1}{2} B_{11} \text{ Var}(\alpha_m/\Gamma_{11}^m)] \qquad (3.2.8)$$

The various terms of (3.2.8) are quite easily rationalized; we have seen them all before. The first represents the positive bias of the efficiency gain that is achieved by guaranteeing that the m^{th} firm now sets marginal

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[18]Recalling Subsection (3.1), we know that the quantity orders and price orders, as well as the price response functions, are invariant across this change in mix. Terms like

$$E\int_{\tilde{q}_i(\text{mix } 1)}^{\tilde{q}_i(\text{mix } 2)} C_1(q_i,\theta_i)dq_i; \; i \neq m$$

cost equal to the price order. As usual, the dampening effect of output variation on expected costs has been included in the term. The last term is similarly the loss registered by the benefit function in response to the newly allowed output variation by the m^{th} firm. Finally, the terms in the sum indicate the dampening/amplifying effects of the negative/ positive correlations in output of the m^{th} firm under price control with the other (n-m) firms already under price control. We should note, in passing, that (3.2.8) is also the precise expression for the comparative advantage of prices over quantities for the m^{th} firm considered alone in the context of its position in the industry as otherwise organized by mix 1.

As would be expected, reversing the switch produces a very similar result. Consider a third mix in which the $(m+1)^{th}$ firm is switched to quantity control. The comparative advantage of mix 3 over the standard, mix 1, is given by

$$\Delta(\text{mix 3/mix 1}) = -\frac{1}{n}\Big[\frac{1}{2}\,\Gamma_{11}^{m+1}\text{Var}\Big(\frac{\alpha_{m+1}}{\Gamma_{11}^{m+1}}\Big)-B_{11}\text{Var}\Big(\frac{\alpha_{m+1}}{\Gamma_{11}^{m+1}}\Big)-\text{Cov}\Big(\frac{\alpha_{m+1}}{\Gamma_{11}^{m+1}};\beta\Big)$$

$$-\,B_{11}\sum_{i=m+2}^{n}\text{Cov}\Big(\frac{\alpha_{m+1}}{\Gamma_{11}^{m+1}};\frac{\alpha_i}{\Gamma_{11}^{i}}\Big)\Big]. \qquad (3.2.9)$$

The only substantive change is the minus sign that converts losses into gains and gains into losses; but Δ(mix 3/mix 1) reflects a comparative of quantities over prices, so such transformations are expected. The content remains otherwise the same and we have demonstrated an intuitively satisfying generalization of Proposition 2:

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are therefore all zero and do not appear in the final equation. The covariance of benefits and output variation under prices at the m^{th} firm does appear, however, and registers the "correctness" of that output variation vis a vis changes in benefits.

Proposition 3:

In order to guarantee the profitability of an alteration in the control mix, it is sufficient to show the existence of one firm for which the opposite mode of control is preferred when that firm is considered individually, but in the context of the industry as otherwise controlled by the original mix.

Several remarks are in order. By setting m equal to zero or n, we have sufficient conditions for the existence of a mix that is preferred to uniform control by prices or quantities, respectively. In addition, even though the existence of an optimal mix can be insured,[19] simply changing the mode of control on all firms that so prefer on an individual basis need not yield that global maximum. It is quite possible, for instance, that making a switch in control with a negative comparative advantage could set up a covariance structure that would ultimately attain a higher value of benefits minus costs.

To see this point, we begin with a mix of m firms under quantities ($m \geq 2$) and (n-m) firms under prices and suppose that there exist no firms for which a change of control would be preferred on an individual basis. We further postulate the existence of a subgroup of p firms, to be numbered from (m-p+1) to m, under quantity control but for which price control would be preferred when they are considered as a subgroup in the context of their cumulative position in the industry as otherwise

[19] We have a finite number of possible mixes whose evaluations in expected benefits minus costs constitute a closed and bounded subset of the real line. Thus, that subset contains its own maximal element.

regulated by the given mix. If we can show that these two statements can be mutually consistent, we will have illustrated our caveat. Notice, too, that we will have also proven that the sufficient conditions recorded in Proposition 3.2 are not necessary. Our first assumption requires that

$$E\left(\frac{r_{11}^{k}}{2}\left(\frac{\alpha_k}{r_{11}^{k}}\right)^2 + \frac{B_{11}}{2n}\left(\sum_{\substack{i=m+1 \\ i\neq k}}^{n}\left(\frac{\alpha_k}{r_{11}^{k}}\right)\left(\frac{\alpha_i}{r_{11}^{i}}\right) + \left(\frac{\alpha_k}{r_{11}^{k}}\right)^2\right) \equiv G(k) < 0$$

$$k = (m-p+1, \ldots, m). \qquad (3.2.10a)$$

Our second assumption similarly requires that

$$\sum_{k=m-p+1}^{m}\{(G(k)) + \frac{B_{11}}{2n}\left(E\sum_{\substack{j=m-p+1 \\ j\neq k}}^{m}\left(\frac{\alpha_k}{r_{11}^{k}}\right)\left(\frac{\alpha_j}{r_{11}^{j}}\right)\right)\} \qquad (3.2.10b)$$

It is quite possible for $G(k)$ to be negative for all k, as prescribed by (3.2.10a), and for the total value of (3.2.10b) to be positive. All that is required is a sufficiently negative correlation of costs within the given subgroup of firms. We should therefore not be lulled into looking at only one firm subgroups by the dependence of our propositions on simple, single firm conditions.

3.2.5: Large Firms and a Preferred Mix

When we envision an industry, we typically think of a collection of large and small firms producing the same output. It is therefore reasonable to ask what influence the relative size of a firm exerts on the choice of control in a preferred mix. We propose two methods of introducing a large firm into the current analysis, in response to that query. Each method involves viewing such a firm as a collection of highly correlated production units. On the one hand, a single cost function for a large firm can be determined by the horizontal addition of the cost curves of many smaller production units; we thereby create a cost curve with a smaller curvature than any of the single units. In the context of this notion, then, we suggest the representation of a large firm by a cost function with a small value for C_{11}. On the other hand, we can preserve the individual production units by defining a large firm to be a collection of perfectly correlated production units with values of C_{11} more in line with the small firms.

Under either interpretation, we can now show that large firms are more likely to be regulated by quantities than prices in a preferred control mix. If we view such a firm as a collection of perfectly correlated units that must face the same control, for instance, we have the following expression for the comparative advantage of prices for that firm in the context of an otherwise arbitrary mix[20]

$$\sum_{k=m-p+1}^{m} \{\frac{1}{2} \Gamma_{11}^{k} \operatorname{Var}(\frac{\alpha_k}{\Gamma_{11}^{k}}) + \frac{1}{2n} B_{11}[\sum_{\substack{i=m+1 \\ i\neq k}}^{n} \operatorname{Cov}(\frac{\alpha_k}{\Gamma_{11}^{k}}; \frac{\alpha_i}{\Gamma_{11}^{i}}) + \operatorname{Var}(\frac{\alpha_k}{\Gamma_{11}^{k}})]$$

$$+ \frac{B_{11}}{2n}[\sum_{\substack{j=m-p+1 \\ j\neq k}}^{m} \operatorname{Cov}(\frac{\alpha_k}{\Gamma_{11}^{k}}; \frac{\alpha_j}{\Gamma_{11}^{j}})] + \operatorname{Cov}(\frac{\alpha_k}{\Gamma_{11}^{k}};\beta)\} \qquad (3.2.11)$$

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[20]The precise case that generates this comparative advantage is an industry of n units, p of which form the firm in question. We have numbered the units so that in (3.2.11), the first (m-p) units face quantities, the last (n-m) face prices, and the middle p are the subject of the discussion.

Since we require that the units within the firm be perfectly correlated, we see that (3.2.11) has a large negative term in its very heart. This negative bias would be extremely difficult to overcome unless the production units either face cost functions with low values of C^i_{11}, or are extremely negatively correlated with other firms under price control. There is no compelling reason to believe that either condition is very likely to occur. It is therefore expected that large firms would face quantity regulation.

We can alternatively view a large firm as one unit with a very small value of C_{11}. The comparative advantage of prices for the firm in the context of an arbitrary mix is again shackled with a negative bias that is only overcome by negative cost covariance with the other firms under price control. As we noted in the previous paragraph, there is no particular reason to predict such correlation, and quantity control still emerges more likely. We are reminded, however, that the large negative bias that we have noted in both arguments are not sufficient to preclude a preferred mix in which the large firm faces price regulation. In further defense of our conclusion, however, also recall that, as demonstrated in the previous subsection, the counterexamples to sufficiency also depend upon a structure of negative output correlations within a subset of the industry. We have already rejected this condition as improbable.

Section 3.3: Output Variation under Quantity Control

Two of the major sources of uncertainty that were introduced in Chapter 2 have been neglected thus far in Chapter 3. The present section will begin to correct for this omission by recording the effects of random variation around quantity orders issued by the center in the n-firm case. ξ_i will index the random variable (s) influencing the i^{th} firm, and $\phi_i(\xi_i)$ will register its effect on the output of that firm. Actual output of the i^{th} firm under an optimal quantity order, $\hat{q}_{ai}$, is therefore additively related to the quantity ordered, $\hat{q}_{pi}$, as follows:

$$\hat{q}_{ai} = \hat{q}_{pi} + \phi_i(\xi_i).$$

The random variable ξ_i is also expected to influence the cost function of the i^{th} firm, so that $C^i(q_i,\theta_i,\xi_i)$ now represents these costs. We presume, in addition, that the random variables are all jointly distributed.

3.3.1: The Center Working with the Correct Distribution

We appeal to Section 3.1 and assert the existence of points $\hat{q}_{oi}(i = 1,\ldots,n)$ such that

$$EC_1^i(\hat{q}_{oi},\theta_i,\xi_i) = EB_1(\hat{q},n)$$

for all i and where $\hat{q}$ is defined to be the sum of the $\hat{q}_{oi}$. The cost function $C^i(q_i,\theta_i,\xi_i)$ is then expanded around $\hat{q}_{oi}$; under our usual assumptions,

$$C^i(q_i,\theta_i,\xi_i) = a_i(\theta_i,\xi_i) + (C_i' + \alpha_i(\theta_i,\xi_i))(q_i-\hat{q}_{oi}) + \frac{1}{2} C_{11}^i(q_i-\hat{q}_{oi})^2,$$

where $C_i' = EC_1^i(\hat{q}_{oi},\theta_i,\xi_i)$ and $\alpha_i(\theta_i,\xi_i) = C_1^i(\hat{q}_{oi},\theta_i,\xi_i) - C_i'$. Benefits are similarly expanded around $\hat{q}$:

$$B(q,n) = b(n) + (B' + \beta(n))(q-\hat{q}) + \frac{1}{2} B_{11} (q-\hat{q})^2,$$

with B' and $\beta(\eta)$ having analogous definitions. Observe once again that for all i, $E\beta(\eta) = E\alpha_i(\theta_i,\xi_i) = 0$ and $B' = C_i'$. Given these approximations, we can begin to compute the comparative advantage of prices over quantities.

There exists an efficiency loss for any set of quantity orders issued by the center, $\bar{q}_{pi}$ (i = 1, ..., n), defined by

$$L(\bar{q}_{pi};\bar{\theta}_1, \ldots, \bar{\theta}_n, \bar{\xi}_1, \ldots, \bar{\xi}_n, \bar{\eta}) = E\left[-\int_{\sum_{i=1}^{n}(\bar{q}_{pi}+\phi_i)}^{\sum_{i=1}^{n} q_i^{opt}} B_1(q,\bar{\eta})\, dq + \sum_{i=1}^{n} \int_{\bar{q}_{pi}+\phi_i}^{q_i^{opt}} C_1(q_i,\bar{\theta}_i,\bar{\xi}_i)dq_i\right.$$

for arbitrary values of the random variables indicated by bars. The first order conditions for the minimization of these losses are

$$E\, B_{11}\left(\sum_{i=1}^{n}(\hat{q}_{pi} + \phi_i(\xi_i) - \hat{q}_{oi})\right) = E\, C_{11}^{i}(\hat{q}_{pi} + \phi_i(\xi_i) - \hat{q}_{oi});$$

$$\forall i = 1, \ldots, n,$$

so that the optimal quantity orders are given by

$$\hat{q}_{pi} = \hat{q}_{oi} - E\phi_i(\xi_i) \tag{3.3.1}$$

for all i = 1, ..., n. The addition of an output distortion under quantities has no effect on the price mode, so that the optimal price order, $\tilde{p}$, and the response function of each firm to that order, $\tilde{q}_i(\theta_i,\xi_i)$ remain unchanged:

$$\tilde{p} = B' = C_i', \text{ for all } i, \text{ and} \tag{3.1.6}$$

$$\tilde{q}_i(\theta_i,\xi_i) = \hat{q}_{oi} - \left(\frac{\alpha_i(\theta_i,\xi_i)}{C_{11}^i}\right). \tag{3.1.7}$$

The comparative advantage of prices over quantities is therefore

$$\begin{aligned}\Delta_{2n} = {} & [\tfrac{1}{2} B_{11} \sum_{i=1}^{n} \text{Var}(\alpha_i/C_{11}^i) + B_{11} \sum_{i=1}^{n} \sum_{j=i}^{n} \text{Cov}(\alpha_i/C_{11}^i;\ \alpha_j/C_{11}^j)] \\ & + [\tfrac{1}{2} \sum_{i=1}^{n} C_{11}^i \text{ Var}(\alpha_i/C_{11}^i)] \\ & + [\sum_{i=1}^{n} \text{Cov}(\phi_i;\alpha_i) - \sum_{i=1}^{n} \text{Cov}(\phi_i;\beta) + \sum_{i=1}^{n} \text{Cov}(\alpha_i/C_{11}^i;\beta)] \\ & - [\tfrac{1}{2} B_{11} \sum_{i=1}^{n} \text{Var } \phi_i + B_{11} \sum_{i=1}^{n} \sum_{j=i}^{n} \text{Cov}(\phi_i;\phi_j)] \\ & + [\tfrac{1}{2} \sum_{i=1}^{n} C_{11}^i \text{ Var } \phi_i]\end{aligned} \tag{3.3.2}$$

Observe that the spirit of Chapter 2 is preserved in (3.3.2). Total output is inserted into the benefit function. The loss in expected benefits created by variation in total output, over the case in which the mean output is produced with certainty, is algebraically equal to $(\frac{1}{2})B_{11}$ times the variance of this distortion in <u>total</u> output. The variance of total output under price regulation is

$$\sum_{i=1}^{n} \text{Var}(\alpha_i/C_{11}^i) + 2 \sum_{i=1}^{n} \sum_{j=i}^{n} \text{Cov}(\alpha_i/C_{11}^i;\alpha_j/C_{11}^j) = \text{Var}(\sum_{i=1}^{n} (\alpha_i/C_{11}^i)),$$

so that the first term in (3.3.2) registers the loss in expected benefits due to output variation under prices. The variance of total output under quantities is similarly

$$\sum_{i=1}^{n} \text{Var } \phi_i + 2 \sum_{i=1}^{n} \sum_{i=j}^{n} \text{Cov}(\phi_i;\phi_j) = \text{Var}(\sum_{i=1}^{n} \phi_i),$$

and the fourth term in (3.3.2) registers the loss in expected benefits under quantities. On the cost side, however, the total increase in expected costs is simply the sum of the increases at the individual firms. The last term is therefore the increase in expected costs caused by output variation under quantity control. The second term in (3.3.2), meanwhile, registers the combined effects of the increase in costs due to output variation under prices and the efficiency gains afforded price controls by the equalization of marginal costs across firms (equal to $\tilde{p}$). Notice that the efficiency gains always dominate; the second term is always positive. The reader can easily check that the signs of the other terms just mentioned are consistent with the definition of Δ_{2n} as the comparative advantage of prices. We finally note that the third term of (3.3.2) registers the amplifying/dampening effects on total output variation of the simultaneous changes in output under both modes and the marginal cost and benefit functions. A similar term appeared in the one firm case and is fully rationalized in the text of Subsection 2.2.1.

As we now correct Δ_{2n} for the pure effect of the number of firms, we perform a familiar transformation on the various cost functions. Define

$$\Gamma^i (x_i,\theta_i,\xi_i) = nC^i (x_i/n,\theta_i,\xi_i);$$

that is, Γ^i represents total costs as a function of total output, x_i, under the assumption that all firms are exact duplicates of the i^{th}. The properties of this transformation are recorded in Section 3.1; the crucial property for our purposes is that

$$\mathrm{Var}(\alpha_i/C^i_{11}) = (1/n^2)\mathrm{Var}(\alpha_i/\Gamma^i_{11})$$

is the variance of <u>total</u> output under the identical firm assumption. To parallel this cost transformation, we define

$$\Phi_i(\xi_i) \equiv n\phi_i(\xi_i)$$

to be the distortion of total output under quantities if all firms are independent and duplicate the i^{th}; as a result,

$$\mathrm{Var}(\phi_i) = (1/n^2)\mathrm{Var}(\Phi_i)$$

and

$$\mathrm{Cov}(\phi_i;\phi_j) = (1/n^2)\mathrm{Cov}(\Phi_i;\Phi_j).$$

We can therefore rewrite equation (3.3.2) as follows:

$$\Delta_{2n} = \left(\frac{B_{11}}{2n^2}\right)\left(\sum_{i=1}^{n} \mathrm{Var}(\alpha_i/C^i_{11}) + 2\sum_{i=1}^{n}\sum_{j=i}^{n} \mathrm{Cov}(\alpha_i/\Gamma^i_{11};\alpha_j/\Gamma^j_{11})\right)$$

$$+ \sum_{i=1}^{n}\left(\frac{\Gamma^i_{11}}{n}\right)\mathrm{Var}(\alpha_i/\Gamma^i_{11}) + \sum_{i=1}^{n}\left[\left(\frac{1}{n}\right)\mathrm{Cov}(\Phi_i;\alpha_i)\right.$$

$$\left. - \left(\frac{1}{n}\right)\mathrm{Cov}(\Phi_i;\beta) + \left(\frac{1}{n}\right)\mathrm{Cov}(-\alpha_i/\Gamma^i_{11};\beta)\right]$$

$$- \left(\frac{B_{11}}{2n^2}\right)\left(\sum_{i=1}^{n} \mathrm{Var}\,\phi_i + 2\sum_{i=1}^{n}\sum_{j=i}^{n} \mathrm{Cov}(\phi_i;\phi_j)\right)$$

$$+ \sum_{i=1}^{n} \Gamma^i_{11}\,\mathrm{Var}\,\Phi_i \qquad (3.3.2)'$$

The effect of a <u>ceteris paribus</u> increase in the number of firms on the comparative advantage of prices can now be demonstrated. We temporarily assume that all of the firms are identical, so that our full attention will be focused on that effect; in particular, allow that $\Gamma^i_{11} = \Gamma_{11}$ and $\Phi_i = \Phi$ for all i = 1, ..., n. Suppose further that

$$\text{Var}\ (\alpha/\Gamma_{11}) = \sigma^2;\ i = 1, \ldots, n;$$
$$\text{Cov}\ (\alpha/\Gamma_{11};\alpha/\Gamma_{11}) = \rho\sigma^2;\ i \neq j \text{ and } i \ \&\ j = 1, \ldots, n;$$
$$\text{Var}\ (\Phi) = \bar{\sigma}^2; \text{ and}$$
$$\text{Cov}\ (\Phi_i;\Phi_j) = \bar{\rho}\bar{\sigma}^2 \text{ across all firms.}$$

The comparative advantage can then be rewritten as

$$\Delta_{2n} = \rho\left(\frac{B_{11}\sigma^2}{2} + \frac{\Gamma_{11}\sigma^2}{2}\right) + (1-\rho)\left(\frac{B_{11}\sigma^2}{2n} + \frac{\Gamma_{11}\sigma^2}{2}\right)$$
$$+ \bar{\rho}\left(\frac{-B_{11}\bar{\sigma}^2}{2} + \frac{\Gamma_{11}\bar{\sigma}^2}{2}\right) + (1-\bar{\rho})\left(-\frac{B_{11}\bar{\sigma}^2}{2n} + \frac{\Gamma_{11}\bar{\sigma}^2}{2}\right)$$
$$+ \text{Cov}(\alpha_i;\Phi) - \text{Cov}(\beta;\Phi) + \text{Cov}(-\alpha/\Gamma_{11};\beta).$$

Notice that only two firms are influenced by an increase in n; their combined effect is recorded below:

$$\frac{1}{n}\left(\frac{1}{2}B_{11}\left((1-\rho)\sigma^2 - (1-\bar{\rho})\bar{\sigma}^2\right)\right).$$

An increase in n will decrease the absolute magnitude of this term. If it is negative, for instance, the entire expression is a negative bias against prices that is diminished as n increases. Observe that it will tend to be negative when the variance of total output under prices is large and when the correlation coefficient across marginal costs is small, or even negative. Both of these conditions work to create circumstances in which there are large diversification gains to be gleaned under prices as n increases. $(\frac{1}{n})(\frac{1}{2} B_{11}((1-\rho)\sigma^2 - (1-\bar{\rho})\bar{\sigma}^2))$ will also tend to be negative when the variance of total output under quantities is small and the correlation across the output disturbances under quantities is large. These conditions indicate that the potential diversification gains under quantity control are small. If the term in question is positive, on the other hand, it is a positive bias for prices that is also diminished as the number of firms grows. Conditions are then such that diversification gains under quantities stand to be larger than those under prices. Notice, finally, that these effects are registered only through the benefit function, since costs are among the things being held constant.

The profitability of policy mixes is the final topic of this subsection; we will operate in the same general context that we used in Section 3.2. Before doing so, however, we refer to the argument recorded in Subsection 3.2.1 to conclude, by parallel reasoning, that even under an arbitrary mix,

$$\hat{q}_{pi} = \hat{q}_{oi} - E\phi_i(\phi_i),$$

$$\tilde{q}_i(\theta_i,\xi_i) = \hat{q}_{oi} - \left(\frac{\alpha_i(\theta_i,\xi_i)}{C^i_{11}}\right), \text{ and}$$

$$\tilde{p} = B'.$$

The base mix is the same as before: the first m firms are regulated by quantities and the remaining (n-m) face price controls. We will consider the advisability of switching the m^{th} firm to price controls (mix 2). The comparative advantage of mix 2 over the base mix, mix 1, is[21]

$$\Delta(\text{mix } 2/\text{mix } 1) = \frac{1}{2}(B_{11}+C_{11}^{m})\,\text{Var}(\alpha_m/C_{11}^{m}) - \frac{1}{2}(B_{11}-C_{11}^{m})\,\text{Var}\,\phi_m$$

$$+ \text{Cov}(\phi_m;\alpha_m) - \text{Cov}(\phi_m;\beta) + \text{Cov}(-\alpha_m/C_{11}^{m};\beta)$$

$$+ \frac{1}{2}B_{11}\left[\sum_{i=m+1}^{n}\text{Cov}(\alpha_m/C_{11}^{m};\alpha_i/C_{11}^{i}) - \sum_{i=1}^{m-1}\text{Cov}(\phi_m;\phi_i)\right.$$

$$\left. - \sum_{i=m+1}^{n}\text{Cov}(\phi_m;\alpha_i/C_{11}^{i}) + \sum_{i=1}^{m-1}\text{Cov}(-\alpha_m/C_{11}^{m};\phi_i)\right] \qquad (3.3.3)$$

If the output distortions affecting the m^{th} firm under both modes of control are independent of the output disturbances of all of the other firms, then (3.3.3) is simply the modified comparative advantage of prices for the m^{th} firm, given its position in the industry. As we will see, however, the specification of a firm's place in the industry is altered drastically if the independence assumption is invalid. The effects of simultaneous changes in the output of the m^{th} firm under quantities with the outputs of the first (m-1) firms under quantity control and the last (n-m) firms under price control are now irrelevant since the m^{th} firm is now facing price control; the term

$$\frac{1}{2}B_{11}\left[\sum_{i=1}^{m-1}\text{Cov}(\phi_m;\phi_i) + \sum_{i=m+1}^{n}\text{Cov}(\phi_m;\alpha_i/C_{11}^{i})\right]$$

is therefore subtracted from the base statistic. The effects of the simultaneous changes in output of the m^{th} firm under price control with

[21]Equation (3.3.3) is derived from integrals of the same form as those designated in the computation of equation (3.2.8).

the outputs of the first (m-1) firms under quantities and the last (n-m) firms under prices are similarly created by the potential control switch at the m^{th} firm. The expression

$$\frac{1}{2} B_{11} \left[\sum_{i=m+1}^{n} \mathrm{Cov}(\alpha_m/C_{11}^m;\alpha_i/C_{11}^i) + \sum_{i=1}^{m-1} \mathrm{Cov}(\alpha_m/C_{11}^m;\phi_i)\right]$$

is then added to the base statistic. By noting these covariances, we have essentially expanded the notion of a firm's place in the industry to include the cross effects with the outputs of the other firms as otherwise regulated by the first mix.

3.3.2: The Center Working with an Inaccurate Distribution

The supporting mathematics of the study is now altered slightly so that we can more easily handle the inaccurate subjective distribution of the center. The shapes of the cost and benefit functions under all states of nature guarantee the existence of points ℓ_{oi} (i = 1, ..., n) such that

$$\hat{E}\, B_1 \left(\sum_{i=1}^{n} \ell_{oi},\eta\right) = \hat{E}\, C_1 (\ell_{oi},\theta_i,\xi_i)$$

for all i.[22] Recall that the hat notation over the expected value operator indicates a computation using the center's inaccurate distribution (see section 2.4). Each cost curve is then expanded around the corresponding ℓ_{oi} and can be represented, under the usual assumptions, as

$$C^i(q_i,\theta_i,\xi_i) = \bar{a}_i(\theta_i,\xi_i) + (\bar{C}_1^i + A_i(\theta_i,\xi_i))(q_i-\ell_{oi}) + \frac{1}{2}\bar{C}_{11}^i(q_i-\ell_{oi})^2.$$

[22]The existence of the ℓ_{oi} is justified in exactly the same manner as the $\hat{q}_{oi}$ were justified in 3.2.1. Recall that $\hat{E}$(---) represents the expected value computed by the center by using the incorrect distribution $\hat{f}$.

The benefit function is meanwhile expanded in like manner around the point $\hat{x}_o \equiv \sum_{i=1}^{n} \hat{x}_{oi}$:

$$B(q,\eta) = \bar{b}(\eta) + (\bar{B}' + \bar{\beta}(\eta))(q-\hat{x}_o) + \frac{1}{2}\bar{B}_{11}(q-\hat{x}_o)^2.$$

These new representations will make the otehrwise difficult first order conditions of the center's maximization procedures quite manageable.

For any set of quantity orders issued by the center, $\bar{q}_{pi}$ $(i = 1,\ldots,n)$, there exists the standard efficiency loss given by

$$-\int_{\Sigma(\bar{q}_{pi}+\phi_i(\xi_i))}^{\Sigma q_i^{opt}} B_1(q,\eta)dq \quad + \quad \sum_{i=1}^{n} \int_{\bar{q}_{pi}+\phi_i(\xi_i)}^{q_i^{opt}} C_1(q_i,\theta_i,\xi_i)dq_i$$

for arbitrary values of the random variables. The center seeks to minimize what it thinks is the expected value of these losses in selecting the optimal quantity orders, $\hat{q}_{pi}$ $(i = 1, \ldots, n)$. It therefore confronts the first order conditions that

$$\hat{E}(\bar{B}_{11}(\sum_{i=1}^{n}(\hat{q}_{pi} + \phi_i(\xi_i) - x_{oi})) = \hat{E}(\bar{C}_{11}^{i}(\hat{q}_{pi} + \phi_i(\xi_i) - \hat{x}_{oi})).$$

Observe that $\hat{q}_{pi}$ is then defined

$$\hat{q}_{pi} = \hat{x}_{oi} - \hat{E}\phi_i(\xi_i)$$

for all i; the shapes of the benefit and cost curves guarantee the uniqueness of this solution.

The computation of the optimal price order follows a similar pattern. The reaction function of the i^{th} firm to any price order is given by

$$q_i(p,\theta_i,\xi_i) = \ell_{oi} + \frac{p-\bar{C}_i'-A_i(\theta_i,\xi_i)}{\bar{C}_{11}^i} \equiv h^i(p,\theta_i,\xi_i)$$

Observing that the price derivative of this function is nonstochastic, we easily see that the optimal price order, $\tilde{p}$, is implicitly defined by

$$\tilde{p} = \hat{E}\, B_1\,(\sum_{i=1}^{n} h^i(\tilde{p},\theta_i,\xi_i),n)$$

so that $p = \bar{B}' = \bar{C}_i'$, for all i. The quantity response curve of the i^{th} firm to the optimal price is therefore

$$\tilde{q}_i(\theta_i,\xi_i) = \ell_{oi} - \frac{A_i(\theta_i,\xi_i)}{\bar{C}_{11}^i}$$

In support of our procedure of changing the points about which the functions are expanded, we now parenthetically compute the comparative advantage of prices for the 1-firm case, using our new specifications:

$$\Delta_2' = -E\int_{\ell_o - \frac{A(\theta,\xi)}{C_{11}}}^{\ell_o - \hat{E}\phi + \phi(\xi)} (B_1(q,n) - C_1(q,\theta,\xi))dq$$

$$= \frac{\bar{B}_{11}+\bar{C}_{11}}{2}\,(\mathrm{Var}(A/\bar{C}_{11}) + E(A/\bar{C}_{11})^2$$

$$- \frac{\bar{B}_{11}-\bar{C}_{11}}{2}\,(\mathrm{Var}\;\phi + (E\phi - \hat{E}\phi)^2)$$

$$+ (E\phi\cdot A - \hat{E}\phi EA) - (E\phi\cdot\beta - \hat{E}\phi E\beta) + (E(\frac{-A\beta}{\bar{C}_{11}}) - E\beta E(-A/\bar{C}_{11})). \qquad (3.3.6)$$

Observe initially that $(\frac{EA}{\bar{C}_{11}}) = (\frac{EA}{\bar{C}_{11}} - \frac{\hat{E}A}{\bar{C}_{11}})$ is the error made by the center in computing the mean of output variation under prices; the expression $(Var(\frac{A}{\bar{C}_{11}}) - (\frac{EA}{\bar{C}_{11}})^2) = E(\frac{A}{\bar{C}_{11}})^2 = E(\frac{A}{\bar{C}_{11}} - \hat{E}\frac{A}{\bar{C}_{11}})^2$ is the second moment of output variation under prices around the incorrect $\hat{E}(\frac{A}{\bar{C}_{11}}) = 0$. The last term of (3.3.6) is similarly the second moment of $\phi(\xi)$ around $\hat{E}\phi$, while the middle term is $E(\phi-\hat{E}\phi)(A-EA)$.[23] Each expression therefore has the precise interpretation of its counterpart in $\hat{\Delta}_2$; our respecification of costs and benefits has altered absolutely nothing.

The comparative advantage of prices with n firms is more complicated:

$$\hat{\Delta}_{2n} = \frac{B_{11}}{2}\{\sum_{i=1}^{n}\sum_{j=1}^{n}(E(\frac{A_i A_j}{\bar{C}^i_{11}\bar{C}^j_{11}}) + \frac{EA_iEA_j}{\bar{C}^i_{11}\bar{C}^j_{11}})\} + \{\frac{1}{2}\sum \bar{C}^i_{11}(E(\frac{A_i}{\bar{C}^i_{11}})^2 + (\frac{EA_i}{\bar{C}^i_{11}})^2\}$$

$$+ \{\sum_{i=1}^{n}((E\phi_i \cdot A_i)-(\hat{E}\phi_i EA_i)) - \sum_{i=1}^{n}((E\phi_i \cdot \beta)-(\hat{E}\phi_i E\beta)) + \sum_{i=1}^{n}((E\frac{A_i}{\bar{C}^i_{11}}\cdot\beta)-(E\frac{A_i}{\bar{C}^i_{11}}E\beta))\}$$

$$- \frac{1}{2}B_{11}\{\sum_{i=1}^{n}\sum_{j=1}^{n}(E\phi_i\phi_j + (E\phi_i - \hat{E}\phi_i)(E\phi_j - \hat{E}\phi_j))\}$$

$$+ \frac{1}{2}\{\sum_{i=1}^{n}\bar{C}^i_{11}(E\phi_i^2 + (E\phi_i - \hat{E}\phi_i)^2)\} \qquad (3.3.7)$$

The loss in expected benefits created by variation in total output under prices, over the case in which the mean is produced with certainty, is again equal to $B_{11}/2$ times the second moment of this distortion in total output. We should expect, from our past experience with inaccurate subjective distributions, that this moment be computed around the incorrect

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[23] The other terms of the expression are similarly covariances around incorrect means computed by the center.

mean deduced by the center. Observe that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{EA_iA_j}{\bar{C}_{11}^i\bar{C}_{11}^j}+\frac{EA_iEA_j}{\bar{C}_{11}^i\bar{C}_{11}^j}\right)=\sum_{i=1}^{n}E\left(\frac{A_i}{\bar{C}_{11}^i}-\frac{EA_i}{\bar{C}_{11}^i}\right)^2$$

$$+\,2\sum_{i=1}^{n}\sum_{j=i}^{n}E\left(\frac{A_i}{\bar{C}_{11}^i}-\frac{EA_i}{\bar{C}_{11}^i}\right)\left(\frac{A_j}{\bar{C}_{11}^j}-\frac{EA_j}{\bar{C}_{11}^j}\right)$$

$$=E\left(\sum_{i=1}^{n}(A_i/\bar{C}_{11}^i-EA_i/\bar{C}_{11}^i)\right)^2$$

and that EA_k is the error made by the center in evaluating the mean of output variation of the k^{th} firm under prices. The first term, therefore, registers the loss in expected benefits due to variation of total output under price controls measured around the incorrect mean computed by the center; our suspicions were correct. The fourth term is similarly the second moment of variation in total output under quantities measured around $\sum_{i=1}^{n}\hat{E}\phi_i$. The increase in expected costs, on the other hand, is simply the sum of the increases noted at the individual firms, but still measured from the center's view of the mean. Since the optimal quantity orders depend upon that view, the expression

$$\frac{1}{2}\sum_{i=1}^{n}(\bar{C}_{11}^i(E(\phi_i)^2+(E\phi_i-\hat{E}\phi_i))=\frac{1}{2}\sum_{i=1}^{n}\bar{C}_{11}^i(E(\phi_i-\hat{E}\phi_i)^2))$$

is therefore the increase in expected costs under quantity control. The second term in (3.3.7) simultaneously registers this loss under prices and the efficiency gains that accrue because the marginal costs of all firms are set equal to $\bar{p}$. Only the third term remains, but it is familiar. It registers the amplifying/dampening effects on total output variation of the simultaneous changes in output under both modes and the marginal cost and benefit schedules; all of these changes are also measured from their incorrect mean. Equation (3.3.7) is, in summary, the precise analog to (3.3.2) in which variation is measured from the incorrect means computed by the center.

The effect of a ceteris paribus increase in the number of firms can now be studied under the same identical firm assumptions outlined in Subsection 3.3.1. Parallel reasoning reveals that the only terms of the comparative advantage of prices that are influenced by n are

$$\frac{1}{n}\left[\frac{1}{2} B_{11} \left((1-\rho)(E(A-EA)^2) - (1-\bar{\rho})(E(\phi-\hat{E}\phi))\right)\right].$$

The spirit of our previous results is preserved in two ways. First of all, the influence of the relative magnitudes of output variation under the two potential modes of control on the direction of the effect of n on the comparative advantage of prices is exactly as before. The variations are, in addition, measured from the subjective means computed by the center. The text of the previous subsection therefore records interpretations that are perfectly applicable here.

As we finally record the profitability of mixes under the influences of imperfect information, the result should be entirely expected. A firm should be switched from quantity control to price control, for example, if the comparative advantage of prices over quantities for that firm taken individually, but in the context of its position in the industry, is positive. The converse is similarly true. The reader should be able to provide both the intuition behind the final results and the precise expression for the comparative advantage of a switch, thereby verifying that the various terms are perfectly analogous to those recorded in

(3.3.4), but measured around incorrect means.

Section 3.4: The Impact of the Consumption Distortion

The remaining source of uncertainty introduced in Chapter Two is the random distortion between the quantity consumed, q_c, and the quantity actually produced, q_a. Suppose, then, that in the context of the previously described model, the quantity consumed of the actual production of the i^{th} firm is

$$q_{ci} = q_{ai} + \psi_i(\lambda_i),$$

regardless of the type of order sent down by the center. We assume further that all of the random variables are jointly distributed. We are modeling, for instance, the emission of a single pollutant from a variety of sources within a particular geographic region. The precise location of each source will specify the corresponding ψ_i and λ_i, as well as the correlation of λ_i with the other λ_j. The effect of these complications on the comparative advantage of prices will now be studied with and without perfect information at the center.

3.4.1: The Center Working with the Correct Distribution

The cost and benefit functions are once again expanded around the $\hat{q}_{oi}$ $(i = 1, \ldots, n)$ and $\hat{q}$, respectively. For any quantity order, $\bar{q}_{pi}$ $(i = 1, \ldots, n)$, issued by the center and arbitrary values of the random variables, there will once again exist an efficiency loss:

$$- \int_{\Sigma(\bar{q}_{pi} + \phi_i(\bar{\xi}_i) + \psi_i(\bar{\lambda}_i)}^{\Sigma q_i^{opt} + \Sigma\psi_i(\bar{\lambda}_i)} B_1(q,\bar{\eta})dq + \sum_{i=1}^{n} \int_{\bar{q}_{pi} + \phi_i(\bar{\xi}_i)}^{q_i^{opt}} C_1(q_i,\bar{\theta}_i,\bar{\xi}_i)dq_i$$

The center will select the optimal quantity orders by minimizing the expected value of these losses, thereby facing the first order conditions that

$$E(B_{11}(\sum_{i=1}^{n}(\hat{q}_{pi}+\phi_i(\xi_i)+\psi_i(\lambda_i)-\hat{q}_{oi})) = E(C_{11}^{i}(\hat{q}_{pi}+\phi_i(\xi_i)-\hat{q}_{oi})). \quad (3.4.1)$$

The solutions to these equations are not obvious, but they can be computed by the following "backdoor" reasoning. We know that the center will select its optimal price order, $\tilde{p}$, by maximizing expected benefits minus costs. Since the price derivative of the reaction function of any firm, $h_1^i(p,\theta_i,\xi_i)$, is nonstochastic,

$$\tilde{p} = E\ B_1\ (\sum_{i=1}^{n} h^i(\tilde{p},\theta_i,\xi_i),\eta)$$

$$= B' + B_{11}\ (\sum_{i=1}^{n} (\frac{-C_i' + \tilde{p}}{C_{11}^{i}}) + \sum_{i=1}^{n} E\psi_i).$$

The definitions of B' and C_i' imply that $B' = C_i'$ for all i and thus

$$\tilde{p}(1 - B_{11}\ \Sigma\ (1/C_{11}^{i})) = B'(1 - B_{11}\ \Sigma\ (1/C_{11}^{i})) + B_{11}\ \Sigma\ E\psi_i.$$

As a result

$$\tilde{p} = B' + \frac{B_{11} \sum_{i=1}^{n} E\psi_i}{(1-B_{11}\Sigma(1/C_{11}^{i}))}. \quad (3.4.2)$$

In all of the quadratic cases that we have discussed thus far, the optimal price order has also been equal to the value of expected marginal costs evaluated at the output level of any firm under optimal quantity control; that is, in the current case, perhaps

$$\hat{p} = E\, C_1^i\,(\hat{q}_{pi} + \phi_i(\xi),\ \theta_i, \xi_i)$$

for all i. Were this true, the optimal quantity order would be given by

$$\hat{q}_{pi} = \hat{q}_{oi} - E\phi_i + \left[\frac{1}{C_{11}^i}\left(\frac{\sum_{i=1}^{n} B_{11}\, E\psi_i}{(1-B_{11}\Sigma(1/C_{11}^i))}\right)\right]. \qquad (3.4.3)$$

Notice that these potential orders satisfy (3.4.1) exactly; the shapes of the relevant functions guarantee the uniqueness of this solution so that (3.4.3) does indeed record the optimal quantity orders.

Only the benefit function reflects the influence of the consumption distortion directly; the valuation arguments of Chapter Two can therefore be applied to this case. Observe that the means of total output under both modes of control have been translated by

$$\sum_{i=1}^{n}\left(\frac{1}{C_{11}^i}\right)\left(\frac{\sum_{i=1}^{n} B_{11}\, E\psi_i}{1-B_{11}\sum_{i=1}^{n}(1/C_{11}^i)}\right).$$

Since the benefit function is quadratic, Theorem 1 of Section 2.3 immediately predicts that if the λ_i are independent of the other random variables, then the consumption distortion is totally neutral. The only new terms created by the distortion are consequently covariances of the various $\psi_i(\lambda_i)$ with the output disturbances under the two opposing schemes of control.[24] If, for instance, $\psi_i(\lambda_i)$ is positively correlated with some $(\frac{-\alpha_j}{C^j_{11}})$, their combined effect amplifies the variation in total output under price control and constitutes a bias against prices. If $\psi_i(\lambda_i)$ and $\phi_j(\xi_j)$ are positively correlated, on the other hand, there exists a bias against quantities for the same reason. The inclusion of the consumption distortion therefore creates the following new terms in the comparative advantage of prices over quantities:

$$B_{11}(\sum_{i=1}^{n}\sum_{j=1}^{n}[(\mathrm{Cov}(\psi_i(\lambda_i);-\alpha_j/C^j_{11}) - (\mathrm{Cov}(\psi_i(\lambda_i);\phi_j(\xi_j))]). \quad (3.4.4)$$

We conclude, finally, that

$$\Delta_{5n} = \Delta_{2n} + B_{11}\ (\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{Cov}(\psi_i;-\alpha_j/C^j_{11}) - \mathrm{Cov}(\psi_i;\phi_j))). \quad (3.4.5)$$

The influence on the comparative advantage of the covariance terms that have just emerged is independent of the number of firms when Δ_{5n} is corrected for the pure effect of n. The expression

$$\frac{1}{n}\ (\frac{B_{11}}{2}\ ((1-\rho)\sigma^2 - (1-\bar{\rho})\bar{\sigma}^2)$$

determined in Section 3.3 therefore records the influence of the number of firms in this model, as well. The analysis of the previous section remains relevant and complete.

The profitability of mixed policies can also be questioned in the

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[24] The remarks of Section 2.3 concerning third order effects and the neutrality of the consumption distortion may also be applied completely intact.

general context introduced above. Beginning with an arbitrary mix (mix 1) of m firms under quantity control and (n-m) firms under prices, we again consider changing the m^{th} firm to price control. The correlations of the $\psi_i(\lambda_i)$ with $\phi_m(\xi_m)$ would no longer be relevant after such a switch; in their place would be the correlations of the $\psi_i(\lambda_i)$ and $(\frac{-\alpha_m}{C_{11}^m})$. The comparative advantage of the second mix over mix 1 is therefore the expression listed in equation (3.3.3) plus

$$B_{11}\ (\sum_{i=1}^{n}(\mathrm{Cov}(\psi_i;-\alpha_m/C_{11}^m) - \mathrm{Cov}(\psi_i;\phi_m))).$$

The only change in the result discussed in the text subsequent to (3.3.3) is hence the further specification of a firm's position in the industry to include the correlations of the two potential output disturbances of that firm and the various consumption distortions of the other firms.

3.4.2: The Center Working with an Inaccurate Distribution

The final substantive model of this chapter repeats the preceding analysis of the consumption distortion in the case in which the center must determine its optimal orders with imperfect knowledge of the distribution of the random variables. The subjective distribution is again denoted with a hat and we expand costs and benefits around $\hat{x}_{oi}$ and $\hat{x}$, respectively (see Subsection 3.3.2 for the definitions of these points). The quantity orders under both modes of control are computed as in 3.4.1; it is no surprise that

$$\hat{q}_{pi} = \hat{x}_{oi} - \hat{E}\phi_i + [\frac{1}{\bar{C}_{11}^i}\ (\frac{\bar{B}_{11}\ \Sigma\ \hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(1/\bar{C}_{11}^i)}),\ \text{and}$$

$$\tilde{q}_i(\theta_i,\xi_i) = \hat{x}_{oi} - \frac{A_i(\theta_i,\xi_i)}{\bar{C}^i_{11}} + [\frac{1}{\bar{C}^i_{11}} (\frac{\bar{B}_{11}\ \Sigma\ \hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(1/\bar{C}^i_{11})})]$$

for all i.

The effect of the errors made by the center can be computed directly from a remark made in the proof of Theorem 1 from Section 2.3: the change in the expected relative valuation of two disturbances, $d_1(x)$ and $d_2(x)$, due to a translation of both disturbances of L is $v_{11}L(Ed_1-Ed_2)$, where v_{11} is the second derivative of the quadratic valuation function. We will consider the benefit side first.

The mean of total output under price control is

$$\sum_{i=1}^{n} \{(E\phi_i-\hat{E}\phi_i) + [\frac{1}{\bar{C}^i_{11}} (\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma 1/\bar{C}^i_{11}})]\} ,$$

while the corresponding mean under quantity control is

$$\sum_{i=1}^{n} \{- \frac{EA_i}{\bar{C}^i_{11}} + [\frac{1}{\bar{C}^i_{11}} (\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(1/\bar{C}^i_{11}})]\}.$$

Their difference is therefore

$$[(\sum_{i=1}^{n} \frac{-EA_i}{\bar{C}^i_{11}}) - \sum_{i=1}^{n} (E\phi_i-\hat{E}\phi_i)],$$

and both are translated

$$\sum_{i=1}^{n} E\psi_i + \sum_{i=1}^{n} (\frac{1}{\bar{C}^i_{11}} (\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(1/\bar{C}^i_{11})})).$$

The change in the comparative advantage of prices registered through benefits is given by

$$\bar{B}_{11}(\sum_{i=1}^{n} E\psi_i + \sum_{i=1}^{n} (\frac{1}{\bar{C}^i_{11}} (\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(1/\bar{C}^i_{11})}))(\sum_{i=1}^{n} \frac{EA_i}{\bar{C}^i_{11}} - \sum_{i=1}^{n} (E\phi_i-\hat{E}\phi_i))). \quad (3.4.6)$$

The mean of the output of the i^{th} firm under prices is similarly

$$(-EA_i/\bar{C}^i_{11}) + (\frac{1}{\bar{C}^i_{11}}(\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(\ )}));$$ [25]

under quantity control, that mean is

$$(E\phi_i-\hat{E}\phi_i) + (\frac{1}{\bar{C}^i_{11}}(\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(\ )})).$$

Their difference is

$$-EA_i/\bar{C}^i_{11} - (E\phi_i-\hat{E}\phi_i),$$

and since both are translated

$$(\frac{1}{\bar{C}^i_{11}})(\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(1/\bar{C}^i_{11})}),$$

the change in the comparative advantage of prices registered through the i^{th} cost function is

$$\bar{C}^i_{11}(EA_i/\bar{C}^i_{11} - (E\phi_i-\hat{E}\phi_i))(\frac{1}{\bar{C}^i_{11}}(\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(\ )})).$$

The total cost effect is hence

$$\{\sum_{i=1}^{n}(EA_i/\bar{C}^i_{11} - (E\phi_i-\hat{E}\phi_i))\}((\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(\ )})). \qquad (3.4.7)$$

The complete effect on the comparative advantage of prices of the errors made by the center in evaluating the $E\psi_i$ is the sum of (3.4.6) and (3.4.7):

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[25]( ) is notation for $(1/\bar{C}^i_{11})$.

$$\bar{B}_{11}(\Sigma E\psi_i + \Sigma(\frac{1}{\bar{C}^i_{11}}(\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(\ )})+(\frac{\bar{B}_{11}\Sigma\hat{E}\psi_i}{1-\bar{B}_{11}\Sigma(\ )}))(\Sigma(EA_i/\bar{C}^i_{11})-\Sigma(E\phi_i-\hat{E}\phi_i))$$

$$= \bar{B}_{11}(\Sigma E\psi_i-\Sigma\hat{E}\psi_i)(\Sigma(\frac{EA_i}{\bar{C}^i_{11}}) - \Sigma(E\phi_i-\hat{E}\phi_i)). \quad (3.4.8)$$

Recall that the covariance effects noted in 3.4.1 are still present. The net change in the comparative advantage produced by the introduction of the consumption distortion under imperfect knowledge is therefore equation (3.4.8) plus equation (3.4.4):

$$= \bar{B}_{11} [(\sum_{i=1}^{n} \sum_{j=1}^{n} E[\psi_i(-A_j/\bar{C}^j_{11}) - (E\psi_i-\hat{E}\psi_i)(E -A_j/\bar{C}_{11_j})]$$

$$- (\sum_{i=1}^{n} \sum_{j=1}^{n} [(E(\psi_i\phi_j) - E\psi_i E\phi_j) + (E\psi_i-\hat{E}\psi_i)(E\phi_j-\hat{E}\phi_j)]]$$

$$= \bar{B}_{11} [\sum_{i=1}^{n} \sum_{j=1}^{n} E[(\psi_i-\hat{E}\psi_i)(- \frac{A_j}{\bar{C}^j_{11}} + \hat{E}(\frac{-A_j}{\bar{C}^j_{11}}))]$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} E[(\psi_i-\hat{E}\psi_i)(\phi_j-\hat{E}\phi_j)]]. \quad (3.4.9)$$

These same correlations are represented in (3.4.5); they are now simply measured from the incorrect means computed by the center.

Equation (3.4.9) is then totally consistent with our previous conclusions; the incidence of imperfect knowledge of the relevant distribution is simply to change the points around which the inherent variations of the model are measured. Given this interpretation, we conclude immediately that the effect of the number of firms and the profitability of a mixed policy set remains exactly as described in the previous subsection.

Section 3.5: The Automobile Example Extended[26]

We can use the data cited in Section 2.5 to illustrate the two fundamental results of this chapter. Before doing so, however, we will review the parts of that data that would be pertinent to the center's prices-quantities choice for the control of vehicular carbon monoxide emissions, and recall the assumptions that were required to use that data in illustrating the center's analysis of that choice. Figure (3.1) reproduces the graphs of marginal benefits and costs in dollars per year as functions of the percentage reduction of the CO emissions of a typical 1967 automobile (see Figure (2.16)). The relevant curvatures of costs and benefits are approximated by the slopes of these two schedules. In addition, the center faced two sources of output uncertainty. The first was profit motivated variation under prices created by the center's imperfect *ex ante* knowledge of the costs of carbon monoxide emission control. The variance of this output disturbance (σ_π^2) was computed to be 1.1 $(\text{gm./mi.})^2$ at 97% emissions reduction and .3 $(\text{gm./mi.})^2$ at 69% emissions reduction. The second source of output uncertainty was created by production variability and it exists under both modes of control. We assumed that the variance of this output disturbance under quantity control (σ_q^2) was .4 $(\text{gm./mi.})^2$ at all levels of reduction. It was impossible to deduce this variance under price controls (σ_p^2), so sensitivity analysis was performed under the assumption that this random variation was independent of the profit motivated variation. The three

[26] The present discussion will focus upon the desired 70% reduction in carbon monoxide emissions found efficient in Section 2.5.

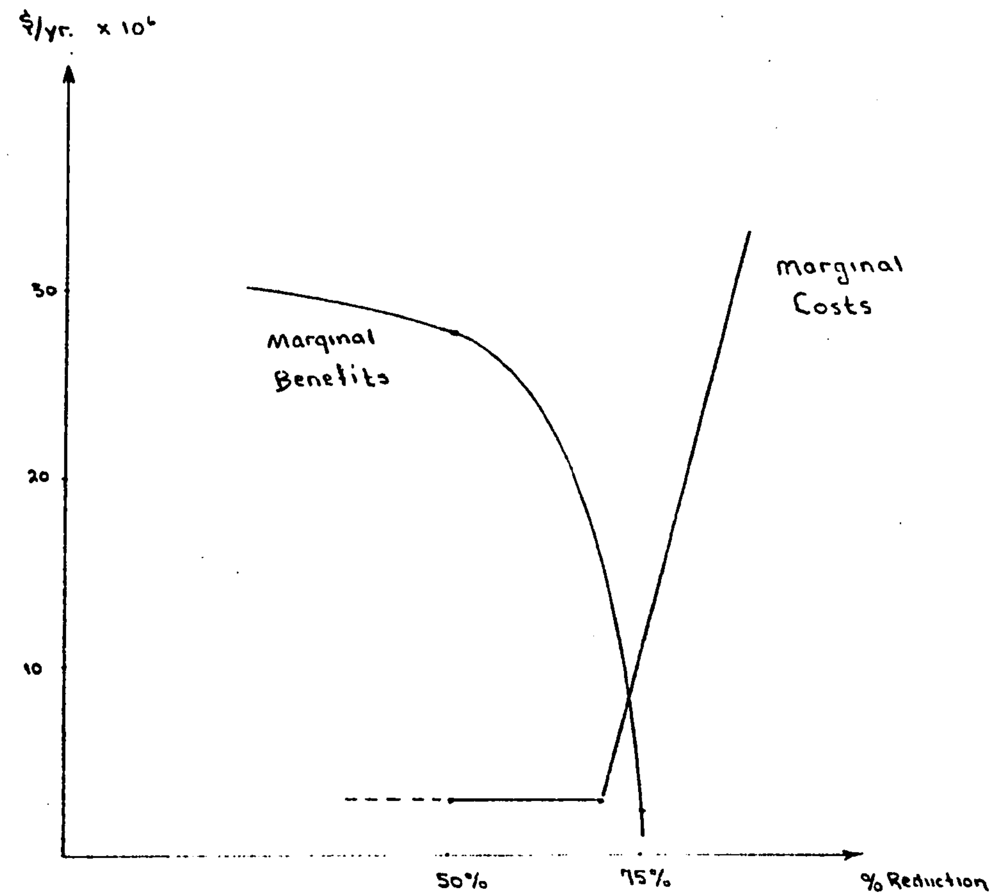
$/yr. x 10⁶
30
20
10
Marginal Benefits
Marginal Costs
50%
75%
% Reduction

FIGURE 3.1

levels assumed were .12, .4, and 1.0 $(\text{gm./mi.})^2$.

Our first concern will be the influence of an increase in the number of automobiles, or equivalently in the number of miles driven per year, on the comparative advantage of prices that the center would compute. Section 3.3 instructs us that the mode of control that is afforded the largest diversification gain by such an increase will be favored by this effect. Recall, however, that this gain may not make this mode the overall better control; it provides only an extra positive bias.

To analyze this effect in our example, we simply need to deduce the correlation across automobiles of the two sources of uncertainty. The profit motivated response to a price order will be applied to each automobile, even though that response is unknown to the center as it decides its order. Profit induced variation can therefore be assumed to be perfectly correlated across cars and the center would perceive no potential for diversification. Meanwhile, each car would be an independent draw from the product variability distribution under either mode. This independence implies that the potential for diversification from this second source of uncertainty under either mode is indicated by the size of the variance in output that each creates. If $\sigma_p^2 = .12$ $(\text{gm./mi.})^2 < \sigma_q^2$, for instance, then quantities offer a larger diversification gain as the number of cars is increased and are therefore afforded a positive bias. If, on the other hand, $\sigma_p^2 = 1.0\ (\text{gm./mi.})^2 > \sigma_q^2$, then prices offer a larger gain and collect the bias. The direction of this effect is thus crucially dependent in this case upon the assumptions we have been forced to make about the size of σ_p^2. The economics behind the effect is, nonetheless, well illustrated.

We can also consider the advisability of mixing controls by regulating some automobiles by prices and others by quantities. To provide a realistic differentiation of the possible groupings, we will assume that larger cars pollute 50% more (in grams per mile) and smaller cars 50% less, ceteris paribus, than does the typical intermediate sized car used to develop the data of Chapter Two. This assumption allows us to demonstrate the potential for a profitable mixed strategy, as well as the importance of the notion of position in an industry. In this context, of course, position in the industry denotes the fraction of the total automobile fleet contained by each particular class of cars; we assume that large cars, small cars, and intermediate cars each constitute about one-third of the current fleet. Figure (3.2) indicates the relevant marginal cost schedules.

Suppose that the center has deemed it most efficient to engineer a 70% average reduction of carbon monoxide emissions. Figure (3.2) reveals that a per unit charge of $\$.4 \times 10^{-5}$/(gm./mi.) per car could achieve this intended reduction. The intermediate sized car, for example, would then be equipped with the 1970 Controlled Combustion System and a PCV valve (system (C)), and be expected to emit 24 grams of carbon monoxide per mile. As we have shown previously, the center would confront a profit motivated variance in emissions equal to $.3\ (\text{gm./mi.})^2$ under this price control. We have also assumed an output variance induced by product variability of $.4\ (\text{gm./mi.})^2$ under the equivalent quantity control; sensitivity analysis must still be performed on the effect of product variability under prices.

Suppose that the profit responses of manufacturers to price controls

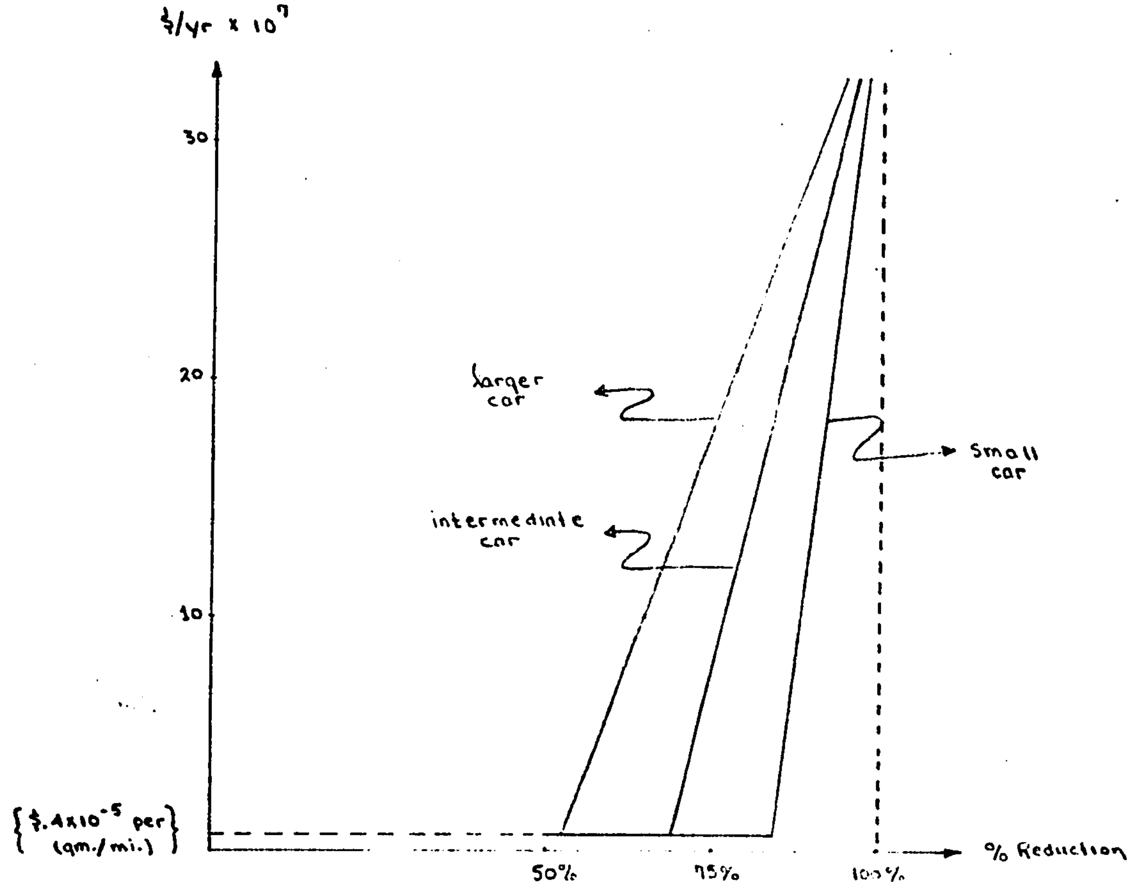
$/yr x 10^7
30
20
10
{$.4x10^-5 per (gm./mi.)}
larger car
intermediate car
small car
50%
75%
100%
% Reduction

FIGURE 3.2

are independent across the size classes of cars. The center must therefore note only that the intermediate class constitutes one-third of the fleet when computing the comparative advantage of prices for that class. As a result, the benefit side of this modified comparative advantage is diminished by a factor of one-third. Applying this observation to equation (2.5.1), we register the modified comparative advantage as follows:

$$(1/2)[((1/3)B_{11}+C_{11})\sigma_\pi^2 + ((1/3)B_{11}-C_{11})(\sigma_p^2 - \sigma_q^2) \qquad (3.5.1)$$

Notice that equation (3.5.1) is quite consistent with equation (3.3.3) that evolved from the theory presented above. Table (3.1) records the sign of this modified computation for the three values of σ_p^2. Comparing

Table 3.1

The Modified Comparative Advantage of Prices at 70% Reduction of CO for Intermediate Sized Cars

$B_{11} = -\$9.0 \times 10^6$/yr. per 1%

$C_{11} = \$1.3 \times 10^7$/yr. per 1%

$\sigma_\pi^2 = .3\ (\text{gm./mi.})^2$

$\sigma_q^2 = .4\ (\text{gm./mi.})^2$

σ_p^2	Δ (Intermediate cars)
.12	$(1/2)((10)(.3) - (22)(.12 - .4)) \times 10^6 > 0$
.4	$(1/2)((10)(.3) - (22)(0)) \times 10^6 > 0$
1.0	$(1/2)((10)(.3) - (22)(1.0 - .4)) \times 10^6 < 0$

Table (3.1) and (2.4), we see that the fractional importance of intermediate sized automobiles in total vehicular carbon monoxide emissions has had no effect on the choice of controls.

When we consider the larger automobiles, however, this fractional importance plays a crucial role. The charge of $\$.4 \times 10^{-5}/(\text{gm./mi.})$ per car will also be met by the installation of system (C) on the larger cars; expected carbon monoxide emissions will be 36 grams per mile (53% reduction over the average 1967 level). To illustrate this case, we assume that product variability under either mode will be the same as before. Figure (3.2), however, suggests a slope of $\$6 \times 10^6$/yr. per 1% for the marginal cost schedule of the larger cars in the neighborhood of 53%. This smaller slope increases the profit motivated variance in emissions facing the center to 1.5 $(\text{gm./mi.})^2$ by increasing the factor that transforms cost deviations into output deviations. Table (3.2) records the modified comparative advantage for the larger cars under these conditions.

Observe that the preferred modes of control remain as before for each value of σ_p^2. Were we to ignore the fractional import of large cars on total carbon monoxide emissions, however, the comparative advantage of prices when $\sigma_p^2 = .12\ (\text{gm./mi.})^2$ would have been

$$(1/2)[(-3.0)(1.5)-(15.0)(.12-.4)] \times 10^6 = (1/2)(-4.5+4.2) \times 10^6 < 0.$$

We would have incorrectly suggested a mixed policy that placed all large cars under quantity controls.

Table 3.2

The Modified Comparative Advantage of Prices at 53% Reduction of CO for Large Cars

$$B_{11} = -\$9.0 \times 10^6/\text{yr. per } 1\%$$
$$C_{11} = \$6.0 \times 10^6/\text{yr. per } 1\%$$
$$\sigma_\pi^2 = 1.5 \text{ (gm./mi.)}^2$$
$$\sigma_q^2 = .4 \text{ (gm./mi.)}^2$$

σ_p^2	Δ (Large cars)
.12	$(1/2)((3.0)(1.5) - (9.0)(.12 - .4)) \times 10^6 > 0$
.4	$(1/2)((3.0)(1.5) - (9.0)(0)) \times 10^6 > 0$
1.0	$(1/2)((3.0)(1.5) - (9.0)(1.0 - .4)) \times 10^6 < 0$

For the sake of completeness, Table (3.3) records the results of similar computations for the smaller cars; they too would be equipped with system (C) in response to a per unit charge of $\$.4 \times 10^{-5}$/(gm./mi.) per car, but would be expected to emit only 12 grams per mile (87% reduction over the average 1967 level). The other statistics are listed in the table. Notice that for the example that we have studied and for any value of σ_p^2, no mixes are preferable to uniform control. There does not exist a single class of automobiles for which the opposite mode of control is preferred when it is considered individually, but in the context of an otherwise uniformly regulated fleet of vehicles.

Table 3.3

The Modified Comparative Advantage of Prices at 87% Reduction of CO for Small Cars

$$B_{11} = -\$9.0 \times 10^6/\text{yr. per } 1\%$$

$$C_{11} = \$2.0 \times 10^7/\text{yr. per } 1\%$$

$$\sigma_\pi^2 = .1 \ (\text{gm./mi.})^2$$

$$\sigma_q^2 = .4 \ (\text{gm./mi.})^2$$

σ_p^2	Δ (Small cars)
.12	$(1/2)((17)(.1) - (23)(.12 - .4)) \times 10^6 > 0$
.4	$(1/2)((17)(.1) - (23)(0)) \times 10^6 > 0$
1.0	$(1/2)((17)(.1) - (23)(1.0 - .4)) \times 10^6 < 0$

Section 3.6: Conclusion

The expansion of the one product case to include multiple producers has begun to demonstrate the strength of the influence of output variation on the prices-quantities comparison. The crucial determinant in the n-firm case is the variance in the total output of the "industry." Variation in the output of each firm can influence only a fraction of total output, and that influence is amplified or dampened by simultaneous variations in the outputs of the other firms. Having thereby taken into account a firm's place in the industry, we have shown that it is profitable to regulate that firm by the mode that would be preferred if it is considered individually, in the context of that position. As the number of firms increases, ceteris paribus, price controls are afforded both an efficiency gain and a diversification gain; quantity controls receive only a diversification gain. The relative magnitude of these two potential gains determines which mode would receive a positive bias from an increase in the number of firms.

Chapter Four

THE REGULATION OF COMPLEMENTS, SUBSTITUTES AND JOINT PRODUCTS

We have thus far concentrated our analysis on the regulation of a single good. In the remainder of the dissertation, we will expand the scope of our study to consider the control of several goods in a second set of circumstances. The present chapter explores the simultaneous regulation of complements and substitutes in consumption, as well as joint products of a single production process. The dissertation will then conclude with a study of intermediate goods in Chapter Five.

Section 4.1: Complements and Substitutes

We begin by specifying a new benefit function that depends on two goods, q_1 and q_2, and a random variable, η, which indexes the uncertainties listed in Chapter One: $B(q_1,q_2,\eta)$. We assume that the two goods are produced separately, and therefore possess their own cost functions: $C^1(q_1,\theta_1,\xi_1)$ and $C^2(q_2,\theta_2,\xi_2)$.[1] We finally presume that η, θ_1, θ_2, ξ_1, and ξ_2 are jointly distributed. The spirit of the previous models has obviously been preserved.

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[1] We also make the standard assumptions about the signs of the derivatives of these functions:

$$C^i_1(q_i,\theta_i,\xi_i) > 0, \text{ and}$$
$$C^i_{11}(q_i,\theta_i,\xi_i) > 0$$

for all (q_i,θ_i,ξ_i) and $i = 1,2$; similarly,

$$B_1(q_1,q_2,\eta) > 0,$$
$$B_2(q_1,q_2,\eta) > 0, \text{ and}$$

The center will maximize expected benefits minus expected costs to determine the optimal production levels for the goods, $\hat{q}_1$ and $\hat{q}_2$. The first order conditions that characterize these optima are:

$$E\,B_1\,(\hat{q}_1,\hat{q}_2,\eta) = E\,C_1\,(\hat{q}_1,\theta_1,\xi_1), \qquad (4.1.1a)$$

$$E\,B_1\,(\hat{q}_1,\hat{q}_2,\ ) = E\,C_2\,(\hat{q}_2,\theta_2,\xi_2), \qquad (4.1.1b)$$

and their existence is guaranteed by the shapes of the curves (see footnote 1). To make the mathematics tractible, we expand the benefit function around $(\hat{q}_1,\hat{q}_2)$, making the usual assumptions about the influence of η:

$$B(q_1,q_2,\eta) = b(\eta) + (B_1'+\beta_1(\eta))(q_1-\hat{q}_1) + \frac{1}{2}B_{11}(q_1-\hat{q}_1)^2$$

$$+ (B'_2+\beta_2(\eta))(q_2-\hat{q}_2) + \frac{1}{2}B_{22}(q_2-\hat{q}_2)^2 + B_{12}(q_1-\hat{q}_1)(q_2-\hat{q}_2). \qquad (4.1.2)$$

where

$$b(\eta) = B(\hat{q}_1,\hat{q}_2,\eta)$$

$$B_1' = E(B_1(\hat{q}_1,\hat{q}_2,\eta))$$

$$\beta_1(\eta) = (B_1(\hat{q}_1,\hat{q}_2,\eta) - B_1')$$

$$B_{11} = B_{11}(\hat{q}_1,\hat{q}_2,\eta)$$

$$B_2' = E(B_2(\hat{q}_1,\hat{q}_2,\eta))$$

$$\beta_2(\eta) = (B_2(\hat{q}_1,\hat{q}_2,\eta) - B_2')$$

$$B_{22} = B_{22}(\hat{q}_1,\hat{q}_2,\eta), \text{ and}$$

$$B_{12} = B_{12}(\hat{q}_1,\hat{q}_2,\eta).$$

$$B_{ii}(q_1,q_2,\eta) < 0$$

for all (q_1,q_2,η) and $i = 1,2$. The sign of $B_{12}(q_1,q_2,\eta)$ depends, of course, on whether good 1 and good 2 are complements or substitutes.

Each cost function is also expanded around its respective optimal quantity:

$$C^i(q_i,\theta_i,\xi_i)=a_i(\theta_i,\xi_i)+(C_i'+\alpha_i(\theta_i,\xi_i))(q_i-\hat{q}_i)+\tfrac{1}{2}C^i_{11}(q_i-\hat{q}_i)^2;\ i=1,2, \tag{4.1.3}$$

where each term has a definition analogous to its counterpart in equation (4.1.2). We note immediately that equations (4.1.1) and (4.1.2) combine with (4.1.3) to imply that

$$B_1' = C_1',\ \text{and} \tag{4.1.4a}$$

$$B_2' = C_2'. \tag{4.1.4b}$$

4.1.1: A First Model--No Output Distortion

We begin our discussion by considering the special case in which a quantity order is produced with certainty. The optimal quantity orders are then

$$\hat{q}_{p1} = \hat{q}_1,\ \text{and} \tag{4.1.5a}$$

$$\hat{q}_{p2} = \hat{q}_2. \tag{4.1.5b}$$

The computation of the optimal price orders is slightly more involved. The price response curve for either firm is

$$h^i(p_i,\theta_i,\xi_i) = \hat{q}_i + \left(\frac{p_i-C_i'-\alpha_i}{C^i_{11}}\right);\ i = 1,\ 2,$$

so that the first order conditions that determine the optimal price orders, $\tilde{p}_1$ and $\tilde{p}_2$, are

$$E\ B_1(h^1,h^2,\eta)\cdot h_1^1 = E\ C_1^1(h^1,\theta_1,\xi_1)\cdot h_1^1; \tag{4.1.6a}$$

$$E\ B_2(h^1,h^2,\eta)\cdot h_1^2 = E\ C_1^2(h^2,\theta_2,\xi_2)\cdot h_1^2. \tag{4.1.6b}$$

Observing that $h_1^i = (1/C_{11}^i)$ and $C_1^i(h^i,\theta_i,\xi_i) = \tilde{p}_i$ $(i = 1,2)$, we note that equations (4.1.6) reduce to

$$E[(B_1'+\beta_1(\eta)+B_{11}(\frac{\tilde{p}_1-\alpha_1-C_1'}{C_{11}^1}) + B_{12}(\frac{\tilde{p}_2-\alpha_2-C_2'}{C_{11}^2})] = \tilde{p}_1; \tag{4.1.6a)'}$$

$$E[(B_2'+\beta_2(\eta)+B_{22}(\frac{\tilde{p}_2-\alpha_2-C_2'}{C_{11}^2}) + B_{12}(\frac{\tilde{p}_1-\alpha_1-C_1'}{C_{11}^1})] = \tilde{p}_2. \tag{4.1.6b)'}$$

The insertion of $\tilde{p}_i = C_i'$ into equations (4.1.6)' causes them to reduce to equations (4.1.4), and the required equality is assured. The uniqueness of this solution is also guaranteed by the shapes of the curves, and we assert that under optimal price control of <u>both</u> goods

$$\tilde{q}_i(\theta_i,\xi_i) = \hat{q}_i - (\frac{\alpha_i(\theta_i,\xi_i)}{C_{11}^i});\ i = 1,2$$

The comparative advantage of placing both goods under price control over placing both goods under quantity control is now available:

$$\Delta_7 = -[E\sum_{i=1}^{2}\int_{\tilde{q}_i}^{\hat{q}_i}((B_i' + \beta_i(\eta) + B_{ii}(q_i-\hat{q}_i))dq_i]$$

$$+[E\sum_{i=1}^{2}\int_{\tilde{q}_i}^{\hat{q}_i}((C_i' + \alpha_i(\theta_i,\xi_i)) + C_{11}^i(q_i-\hat{q}_i)dq_i]$$

$$-E\int_{\tilde{q}_1}^{\hat{q}_1}\int_{\tilde{q}_2}^{\hat{q}_2} B_{12}\ dq_1\ dq_2$$

$$= \frac{1}{2}(B_{11}+C^1_{11})\ \mathrm{Var}(-\alpha_1/C^1_{11}) + \mathrm{Cov}(\beta_1(n);\ -\alpha_1/C^1_{11})$$

$$+ \frac{1}{2}(B_{22}+C^2_{11})\ \mathrm{Var}(-\alpha_2/C^2_{11}) + \mathrm{Cov}(\beta_2(n);\ -\alpha_2/C^2_{11})$$

$$+ B_{12}\ \mathrm{Cov}(-\alpha_1/C^1_{11};\ -\alpha_2/C^2_{11}). \qquad (4.1.7)$$

When $B_{12} = 0$, Δ_7 is simply the sum of the comparative advantage of prices over quantities for each good considered separately. Only the final term, therefore, depends upon the substitutability or complementarity of the two goods. This simple observation confirms the relevance of the individualized analysis of Chapters Two and Three when the good is not subject to cross-consumption effects (i.e., its portion of the overall benefit function is separable).

Goods q_1 and q_2 are complements, however, when $B_{12} > 0$. Indifference curves between the two goods are then highly curved, intuitively suggesting a ratio near which they should be consumed (see Figure 4.1). The output of the two goods tend to move in the same direction under prices when $\mathrm{Cov}(-\alpha_1/C^1_{11};\ -\alpha_2/C^2_{11}) > 0$, thereby preserving this approximate consumption ratio and creating a positive bias for prices. Should that covariance be negative, on the other hand, output levels would tend in opposite directions and the consumption ratio would have to change. The second case causes a bias against prices. Notice that the final term of (4.1.7) reflects these observations.

The rigor behind the above intuition is also easily developed. Suppose that $B_{12} > 0$ and $\mathrm{Cov}(-\alpha_1/C^1_{11};-\alpha_2/C^2_{11}) > 0$. An increase in the output of q_1 will then be typically accompanied by an increase in the output of q_2. By recalling that marginal benefits with respect to q_2

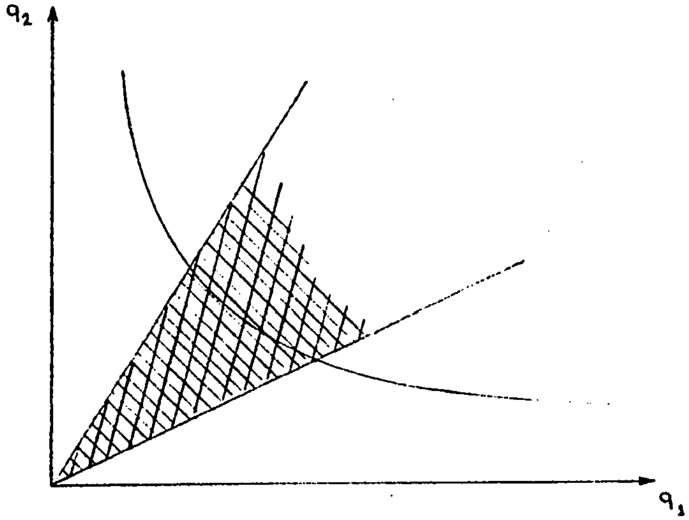

Figure 4.1: Complements

The consumption point should remain roughly within the shaded cone.

are of the form $(B' + \beta(\eta) + B_{22}(q_2-\hat{q}_2) + B_{12}(q_1-\hat{q}_1)$, we can observe that an increase in the marginal benefit of q_2 is simultaneously induced by such an increase in q_1. The production of q_2 therefore tends to increase just when its marginal benefit increases. A positive bias for prices is thus recorded, since the induced effect is in the correct direction. The direction is also correct when B_{12} and the covariance are both negative. Goods q_1 and q_2 are then substitutes, and one should expect that the outputs would optimally move in opposite directions. When only one is negative, however, the direction of the induced effect is incorrect, and a bias against prices similarly recorded.

4.1.2: The Output Distortion

Before we extend our discussion to consider mixed policies, we will repeat the previous analysis in the context of the output distortion.

The quantity actually produced of either good, q_{ai}, is assumed to be additively related to the quantity of that good ordered, q_{pi}, by a random distortion:

$$q_{ai} = q_{pi} + \phi_i(\xi_i);\ i = 1,2.$$

The random variable ξ_i, of course, also appears in the cost functions. The center must therefore confront the following first order conditions in deducing the optimal quantity orders:

$$E\ C_1^1\ ((\hat{q}_{p1}+\phi_1(\xi_1),\ \theta_1,\xi_1)) = E\ B_1\ (\hat{q}_{a1},\hat{q}_{a2},\eta); \qquad (4.1.8a)$$

$$E\ C_1^2\ ((\hat{q}_{p2}+\phi_2(\xi_2),\theta_2,\xi_2)) = E\ B_2\ (\hat{q}_{a1},\hat{q}_{a2},\eta). \qquad (4.1.8b)$$

Observe that

$$\hat{q}_{pi} = \hat{q}_i - E(\phi_i(\xi_i));\ i = 1,2 \qquad (4.1.9)$$

satisfy equations (4.1.8) and are therefore the optimal orders. The computations on the price control side are unaltered, so that the comparative advantage of prices emerges as

$$\begin{aligned}\Delta_8 = \Delta_7 &- \tfrac{1}{2}(B_{11}-C_{11}^1)\text{Var}\ \phi_1 - \text{Cov}\ (\beta_1(\eta);\ \phi_1)\\ &+ \text{Cov}\ (\alpha_1;\phi_1) - \tfrac{1}{2}(B_{22}-C_{11}^2)\ \text{Var}\ \phi_2 - \text{Cov}\ (\beta_2(\eta);\ \phi_2)\\ &+ \text{Cov}\ (\alpha_2;\phi_2) - B_{12}\ \text{Cov}\ (\phi_1,\phi_2). \qquad (4.1.10)\end{aligned}$$

Equation (4.1.10) is still the sum of the comparative advantages of two goods considered alone, when $B_{12} = 0$. Otherwise, the term $B_{12}\text{Cov}(\phi_1;\phi_2)$ works for quantities in precisely the same fashion as did

$B_{12}Cov(-\alpha_1/C^1_{11};-\alpha_2/C^2_{11})$ for prices. Simultaneous movement in ϕ_1 and ϕ_2 creates a positive bias for quantities when that movement is correct with respect to the induced changes in marginal benefits. That positive bias is then subtracted from Δ_7. These covariances become the crucial determinant in the prices-quantities comparison when B_{12} becomes arbitrarily negative or positive, since the induced effects then dominate; Table 4.1 summarizes the results.

Table 4.1

Δ_8 for large $|B_{12}|$*

	$B_{12} \to \infty$	$B_{12} \to -\infty$
Cov I > 0; Cov II < 0	∞	$-\infty$
Cov I > 0; Cov II < 0	$-\infty$	∞
Cov I > Cov II > 0	∞	$-\infty$
0 < Cov I < Cov II	$-\infty$	∞
Cov I < Cov II < 0	$-\infty$	∞
0 > Cov I > Cov II	∞	$-\infty$

*We have defined Cov I $\equiv Cov(-\alpha_1/C^1_{11};-\alpha_2/C^2_{11})$ and Cov II $\equiv Cov(\phi_1;\phi_2)$.

The conclusions listed in Chapter Two concerning the sign of the comparative advantage when the other parameters near their extremes extend to this case intact. They do, however, strongly suggest the advisability of a mix in some circumstances. Output variation under prices disappears for either good as C^i_{11} nears infinity, and prices are therefore preferred. The variances in the output of q_i are meanwhile crucially compared when B_{ii} becomes arbitrarily negative. Suppose, then, that

$\mathrm{Var}(\phi_2) < \mathrm{Var}(-\alpha_2/C^2_{11})$ while C^1_{11} and $|B_{22}|$ approach infinity at the same rate. The sign of Δ_8 will then depend on the sign of $(\mathrm{Var}(\phi_1) - (\mathrm{Var}(-\alpha_2/C^2_{11}) - \mathrm{Var}(\phi_2)))$. Regardless of the control that is deemed preferable by Δ_8, it will be very wrong for one of the two goods. Price regulation of q_1, and quantity regulation of q_2, would seem to be a much better choice. The profitability of such mixes is the subject of the next subsection.

4.1.3: Policy Mixes Across Goods

Suppose that we wish to compare a mix that specifies price regulation of the first good and quantity regulation of the second with quantity control of both. We know that the optimal quantity orders of the second scheme are $\hat{q}_{pi} = \hat{q}_i - E(\phi_i)$. We need only compute the optimal orders and quantity responses under the mixed scheme. The first order conditions that determine the orders are the following:

$$E\,(B_1(h^1(\tilde{p}_1,\theta_1,\xi_1),\,(\hat{q}_{p2}+\phi_2(\xi_2)),\eta)) = \tilde{p}_1;$$

$$E\,(B_2(h^1(\tilde{p}_1,\theta_1,\xi_1),\,(\hat{q}_{p2}+\phi_2(\xi_2)),\eta)) = E\,C^2_1((\hat{q}_{p2}+\phi_2(\xi_2)),\theta_2,\xi_2).$$

Expressing these conditions in terms of our approximations, we can conclude that

$$\tilde{p}_1 = C_1',$$

$$\tilde{q}_1(\theta_1,\xi_1) = \hat{q}_1 - (\alpha_1/C^1_{11}),\ \text{and}$$

$$\hat{q}_{p2} = \hat{q}_2 - E\phi_2;$$

the quantity order on q_2 is invariant to change in the control placed on the first good.

The comparative advantage of the mix over straight quantity control can now be computed:

$$\Delta(pq/qq) = \frac{1}{2}(B_{11}+C_{11}^{1}) \text{ Var } (-\alpha_1/C_{11}^{1}) + \text{Cov } (\beta_1(\eta);-\alpha_1/C_{11}^{1})$$

$$- \frac{1}{2}(B_{11}-C_{11}^{1}) \text{ Var } (\phi_1) - \text{Cov } (\beta_1(\eta);\phi_1) + \text{Cov } (\alpha_1;\phi_1)$$

$$+ B_{12} (\text{Cov}(-\alpha_1/C_{11}^{1};\phi_2) - \text{Cov}(\phi_1;\phi_2). \qquad (4.1.11)$$

It is not surprising that when $B_{12} = 0$, $\Delta(pq/qq)$ is simply the comparative advantage of prices for good 1. The B_{12} terms, of course, reflect the now familiar effect of simultaneous shifts in the outputs of the two goods in the context of induced changes in marginal benefits. $B_{12} \text{ Cov}(-\alpha_1/C_{11}^{1};\phi_2)$ records the effect that would occur if the mix were selected; it is added to the comparative advantage. $B_{12}\text{Cov}(\phi_1;\phi_2)$ records the effect that would occur were both goods regulated by quantities. Since this second effect would be foregone if the mix were selected, it is subtracted. The profitability of this mix is therefore dependent upon the cross effects that are both established and foregone, as well as the comparative advantage of prices for the switched good considered in isolation.

Were we to compare the given mix with a scheme that placed both goods under price controls, a second comparative advantage could be computed:

$$\Delta(qp/pp) = -[\frac{1}{2}(B_{11}+C_{11}^{1})\text{Var}(-\alpha_1/C_{11}^{1}) + \text{Cov}(\beta_1(\eta);-\alpha_1/C_{11}^{1})$$

$$- \frac{1}{2}(B_{11}-C_{11}^{1})\text{Var}(\phi_1) - \text{Cov}(\beta_1;\phi_1) + \text{Cov}(\alpha_1;\phi_1)]$$

$$+ B_{12}[\text{Cov}(\phi_1;-\alpha_2/C_{11}^{2}) - \text{Cov}(-\alpha_1/C_{11}^{1};-\alpha_2/C_{11}^{2})]. \qquad (4.1.12)$$

The terms have the same interpretations, save the first which is predictably the comparative advantage of quantities over prices for good 1 taken alone. The profitability of the mix is still dependent upon both the cross effects and this appropriate isolated comparative advantage.

Notice finally that equations (4.1.11) and (4.1.12) have a second interpretation. $\Delta(pq/qq)$ is also the comparative advantage of prices over quantities for good q_1, given that q_2 is optimally regulated by quantities. $\Delta(qp/pp)$ is similarly the comparative advantage of quantities for q_1, given the optimal price regulation of q_2. This observation suggests a simple generalization: it pays to switch the controls of any good for which the alternative control is preferred in the context of an otherwise constant regulatory scheme.

4.1.4: Conclusions

The previous two chapters list the variances and covariances in output with which the center must be concerned as it imposes control on the production of goods in isolation. When two goods that are substitutes or complements are to be regulated, however, output variation in one good creates an induced effect on the marginal benefits of the second. The center must therefore consider this induced effect in the context of variation in the second good, as well as the other previously mentioned consequences of output fluctuation, in comparing prices and quantities.

With either mode of control, or indeed with any mix, there exists simultaneous variation in both goods. The two goods, in particular, will tend to vary in the same direction, opposite directions, or independently. The expected value of the induced effect is always zero if they vary independently. The desirability of output fluctuation will otherwise depend upon the nature of the two goods; if they are substitutes (complements), the mode of control, or the mix, that on average allows opposite (parallel) variation is favored more. The importance of this effect relative to those listed before depends, of course, on the degree of substitutability or complementarity that is displayed by the two goods in question.

These results are valid regardless of the number of firms producing either good; all of the above comments would be cast in terms of the total output of the firms producing a particular good. An increase in the number of firms producing one of the goods will favor the control that allows the larger reduction in total output variation of that good, just as it did in the third chapter, independent of the control imposed on the other good.

Section 4.2: The Regulation of Joint Products

The simplest way to incorporate joint products into our analysis is to consider a cost function that depends upon two goods, q_1 and q_2, and a vector of random variables $\vec{\theta}$: $C(q_1,q_2,\vec{\theta})$.[2] We require that $\vec{\theta}$ be a vector so that random events that affect the costs of each good individually, as well as those events that affect the costs of both goods,

[2]This cost function has the following characteristics: the first derivatives and the second derivatives with respect to both goods are all positive, and the cross second derivative depends on the nature of the two goods, for all $(q_1,q_2,\vec{\theta})$.

are simultaneously modeled.[3] We continue to assume that the benefit function is $B(q_1,q_2,\eta)$, and that $\vec{\theta}$ and η are jointly distributed. The optimal quantities of the two goods are then determined by maximizing expected benefits minus costs; $\hat{q}_1$ and $\hat{q}_2$ are therefore defined by

$$EB_1(\hat{q}_1,\hat{q}_2,\eta) = EC_1(\hat{q}_1,\hat{q}_2,\vec{\theta}); \quad (4.2.1a)$$

$$EB_2(\hat{q}_1,\hat{q}_2,\eta) = EC_2(\hat{q}_1,\hat{q}_2,\vec{\theta}). \quad (4.2.1b)$$

The assumed shapes of the functions (see footnotes 1 and 2) guarantee the existence and uniqueness of $(\hat{q}_1,\hat{q}_2)$ in this context.

To make the mathematics tractible, we again expand the functions around the point $(\hat{q}_1,\hat{q}_2)$, given our usual assumptions concerning the influences of the random variables. The approximation of the benefit function is still recorded by equation (4.1.2). The new cost function is meanwhile

$$\begin{aligned} C(q_1,q_2,\vec{\theta}) = {} & a(\vec{\theta}) + (C_1'+\alpha_1(\vec{\theta}))(q_1-\hat{q}_1) + \tfrac{1}{2} C_{11} (q_1-\hat{q}_1)^2 \\ & + (C_2' + \alpha_2(\vec{\theta}))(q_2-\hat{q}_2) + \tfrac{1}{2} C_{22} (q_2-\hat{q}_2)^2 \\ & + C_{12} (q_1-\hat{q}_1)(q_2-\hat{q}_2), \end{aligned} \quad (4.2.2)$$

where

$$a(\vec{\theta}) = C(\hat{q}_1,\hat{q}_2,\vec{\theta}),$$

$$C_1' = E(C_1(\hat{q}_1,\hat{q}_2,\vec{\theta})),$$

[3] Any of the random variables could have been vectors; the analysis would obviously not change. The point here is that θ must be a vector, since it will include, at the very least, the variables ξ_1 and ξ_2 that create the output distortions in goods 1 and 2 respectively.

$$\alpha_1(\vec{\theta}) = C_1(\hat{q}_1,\hat{q}_2,\vec{\theta}) - C_1',$$
$$C_{11} = C_{11}(\hat{q}_1,\hat{q}_2,\vec{\theta}),$$
$$C_2' = E(C_2(\hat{q}_1,\hat{q}_2,\vec{\theta})),$$
$$\alpha_2(\vec{\theta}) = C_2(\hat{q}_1,\hat{q}_2,\vec{\theta}) - C_2',$$
$$C_{22} = C_{22}(\hat{q}_1,\hat{q}_2,\vec{\theta}), \text{ and}$$
$$C_{12} = C_{12}(\hat{q}_1,\hat{q}_2,\vec{\theta}).$$

It should be noted in passing that the sign of C_{12} reflects the nature of the two goods in the joint production process. They are substitutes in production if $C_{12} > 0$, and complements if $C_{12} < 0$.

4.2.1: A First Model--No Output Distortion

We begin our analysis of joint products with the simplest case: quantity orders are assumed produced with certainty. The optimal quantity orders are therefore $\hat{q}_1$ and $\hat{q}_2$ by definition. This assumption does not effect the price order, however, and its determination will prove more difficult.

The reaction functions to any price order couple (p_1,p_2), notationally represented by

$$q_i(p_1,p_2,\vec{\theta});\ i = 1,2,$$

are defined implicitly by

$$p_1 = C_1' + \alpha_1(\vec{\theta}) + C_{11}(q_1(p_1,p_2,\vec{\theta})-\hat{q}_1) + C_{12}(q_2(p_1,p_2,\vec{\theta})-\hat{q}_2); \quad (4.2.3a)$$

$$p_2 = C_2' + \alpha_2(\vec{\theta}) + C_{22}(q_2(p_1,p_2,\vec{\theta})-\hat{q}_2) + C_{12}(q_1(p_1,p_2,\vec{\theta})-\hat{q}_1); \quad (4.2.3b)$$

under the assumption that the producer of q_1 and q_2 is a profit maximizer. Equations (4.2.3) can be rewritten in the following form:

$$q_1(p_1,p_2,\vec{\theta}) = \hat{q}_1 + (\frac{p_1-\alpha_1-C_1'}{C_{11}}) - (C_{12}/C_{11})(q_2(p_1,p_2,\vec{\theta})-\hat{q}_2); \quad (4.2.3a)'$$

$$q_2(p_1,p_2,\vec{\theta}) = \hat{q}_2 + (\frac{p_2-\alpha_2-C_2'}{C_{22}}) - (C_{12}/C_{22})(q_1(p_1,p_2,\vec{\theta})-\hat{q}_1). \quad (4.2.3b)'$$

Solving equations (4.2.3)' simultaneously, we conclude that

$$q_1(p_1,p_2,\vec{\theta}) = \hat{q}_1+(C_{11}C_{22}-C_{12}^2)^{-1}(C_{22}(p_1-C_1'-\alpha_1)-C_{12}(p_2-C_2'-\alpha_2)); \quad (4.2.4a)$$

$$q_2(p_1,p_2,\vec{\theta}) = \hat{q}_2+(C_{11}C_{22}-C_{12}^2)^{-1}(C_{11}(p_2-C_2'-\alpha_2)-C_{12}(p_1-C_1'-\alpha_1)). \quad (4.2.4b)$$

We observe in passing that the second order condition for profit maximization requires that

$$\begin{vmatrix} -C_{11} & -C_{12} \\ -C_{12} & -C_{22} \end{vmatrix} = C_{11}C_{22}-(C_{12})^2 = D > 0,$$

and the denominators of equations (4.2.4) are necessarily positive.

The center determines its optimal price orders, $\tilde{p}_1$ and $\tilde{p}_2$, by maximizing expected benefits minus costs with respect to p_1 and p_2 given equations (4.2.4). The first order conditions for this maximization read

$$\tilde{p}_1C_{22} = E(B_1(q_1(\tilde{p}_1,\tilde{p}_2,\vec{\theta}),q_2(\tilde{p}_1,\tilde{p}_2,\vec{\theta}),\eta)\ C_{22}; \quad (4.2.5a)$$

$$\tilde{p}_2C_{11} = E(B_2(q_1(\tilde{p}_1,\tilde{p}_2,\vec{\theta}),q_2(\tilde{p}_1,\tilde{p}_2,\vec{\theta}),\eta)\ C_{11}. \quad (4.2.5b)$$

Equations (4.2.5) are satisfied by

$$p_i = C_i' = B_i';\ i = 1,2;$$[4]

uniqueness is guaranteed as before and the quantity responses to the optimal prices are clearly

$$\tilde{q}_1(\vec{\theta}) = \hat{q}_1+(1/D)(C_{12}\alpha_2(\vec{\theta})-C_{22}\alpha_1(\vec{\theta})),\ \text{and} \tag{4.2.6a}$$

$$\tilde{q}_2(\vec{\theta}) = \hat{q}_2+(1/D)(C_{12}\alpha_1(\vec{\theta})-C_{11}\alpha_2(\vec{\theta})). \tag{4.2.6b}$$

A change in $\vec{\theta}$ will therefore create a change in the amount of q_1 (e.g.) produced under prices by directly influencing the marginal costs of that good; $(-1/D)C_{22}\alpha_1(\vec{\theta})$ reflects this direct effect. There is, in addition, an induced effect created by the following sequence of events. A change in $\vec{\theta}$ also causes the amount of q_2 being produced under prices to change; this influence on q_2 has a secondary effect on the marginal cost of q_1 through C_{12}. The term $(-1/D)C_{12}\alpha_2(\vec{\theta})$ translates this induced effect on marginal costs into a second change in the amount of q_1 produced. The distinction between the direct and indirect effects of $\vec{\theta}$ will be important as we now turn to consider the comparative advantage of price regulation of both goods over quantity regulation of both goods.

The comparative advantage of prices is computed as before, and emerges in the following form:

$$\begin{aligned}\Delta_9 = {} & \frac{1}{2}(B_{11}-C_{11})\mathrm{Var}(\tfrac{1}{D}(C_{12}\alpha_2-\alpha_1C_{22})) + \mathrm{Cov}((\beta_1-\alpha_1);(\tfrac{1}{D}(C_{12}\alpha_2-\alpha_1C_{22}))) \\ & + \frac{1}{2}(B_{22}-C_{22})\mathrm{Var}(\tfrac{1}{D}(C_{12}\alpha_1-\alpha_2C_{11})) + \mathrm{Cov}((\beta_2-\alpha_2);(\tfrac{1}{D}(C_{12}\alpha_1-\alpha_2C_{11}))) \\ & + (B_{12}-C_{12})\mathrm{Cov}((\tfrac{1}{D})(C_{12}\alpha_2-\alpha_1C_{22});(\tfrac{1}{D})(C_{12}\alpha_1-\alpha_2C_{11})).\end{aligned} \tag{4.2.7}$$

[4]The approximations combine with 4.2.1 to guarantee that $B_i'=C_i';i=1,2.$

When we recall the expressions given for the reaction functions to $\tilde{p}_1$ and $\tilde{p}_2$, the only unfamiliar term in Δ_9 is

$$-C_{12}\text{Cov}((\frac{1}{D})(C_{12}\alpha_2-\alpha_1 C_{22});(\frac{1}{D})(C_{12}\alpha_1-\alpha_2 C_{11})).$$

This newcomer is explained most easily by use of an oversimplified example; we will compare the two cases described in Table 4.2.

Table 4.2

The C_{12} Term--An Example

Case	Probability	Output q_1	Output q_2
Case I	1/2	$\hat{q}_1 + L_1$	$\hat{q}_2 + L_2$
	1/2	$\hat{q}_1 - L_1$	$\hat{q}_2 - L_2$
Case II	1/2	$\hat{q}_1 + L_1$	$\hat{q}_2 - L_2$
	1/2	$\hat{q}_1 - L_1$	$\hat{q}_2 + L_2$

The expected cost for Case I is

$$Ea + C_{11}(L_1)^2 + C_{22}(L_2)^2 + C_{12}L_1L_2;$$

for Case II, it is similarly

$$Ea + C_{11}(L_1)^2 + C_{22}(L_2)^2 - C_{12}L_1L_2.$$

Case II is obviously cheaper when C_{12} is positive because of the negative correlation of outcomes. The C_{12} term is therefore a bias against prices when the covariance of the output changes that prices allow is positive. The reverse is, of course, equally true when $C_{12} < 0$.

A term by term interpretation of the cost side of (4.2.7) can now be presented; we will begin with q_1. The increase in expected costs due to output variation in q_1 is recorded by

$$-\frac{1}{2} C_{11} \text{ Var } ((\frac{1}{D}) (C_{12}\alpha_2-\alpha_1 C_{22}))$$

and works in favor of quantities. The term $(-\text{Cov}(\alpha_1;(\frac{C_{12}\alpha_2-\alpha_1 C_{22}}{D})))$ similarly records the efficiency gains or losses created by the variation in the output of q_1 relative to the uninduced changes in marginal costs. Were this covariance positive (e.g.), output of q_1 would tend to increase as marginal costs increase and thereby construct a bias against the price controls that allow such variation. The C_{12} term can be thought to record the analogous gains or losses created by the variation in the output of q_1 relative to the induced changes in marginal costs. The algebraic sum of these three terms can be shown to be

$$+\frac{1}{2} C_{11} \text{ Var } ((1/D) (C_{12}\alpha_2 - \alpha_1 C_{22})),$$

a net positive bias for prices.

A similar story can be told for q_2 and results in a net effect of

$$+\frac{1}{2} C_{22} \text{ Var } ((1/D) (C_{12}\alpha_1 - \alpha_2 C_{11})).$$

Observe, however, that the C_{12} term has then been used twice. When we collect terms to express Δ_9 more succinctly,

$$C_{12} \text{ Cov } ((1/D)(C_{12}\alpha_2-\alpha_1 C_{22}); (1/D)(C_{12}\alpha_1-\alpha_2 C_{11}))$$

must therefore appear with the opposite sign:

$$\Delta_9 = \frac{1}{2}(B_{11}+C_{11})\mathrm{Var}((1/D)(C_{12}\alpha_2-\alpha_1C_{22})) + \mathrm{Cov}\,(\beta_1;(1/D)(C_{12}\alpha_2-\alpha_1C_{22}))$$

$$+ \frac{1}{2}(B_{22}+C_{22})\mathrm{Var}((1/D)(C_{12}\alpha_1-\alpha_2C_{11})) + \mathrm{Cov}\,(\beta_2;(1/D)(C_{12}\alpha_1-\alpha_2C_{11}))$$

$$+ (B_{12}+C_{12})\ \mathrm{Cov}\ ((1/D)(C_{12}\alpha_2-\alpha_1C_{22});\ (1/D)(C_{12}\alpha_1-\alpha_2C_{11})). \qquad (4.2.7)'$$

4.2.2: The Output Distortion

Before discussing the comparative advantage as the relevant parameters approach their extremes, we correct for the asymmetry in cutput variation of our first joint products model by once again introducing the output distortion. The quantity actually produced, q_{ai}, is thus defined by

$$q_{ai} = q_{pi} + \phi_i(\xi_i);\ i = 1,2,$$

where q_{ai} is the quantity ordered by the center. The ξ_i are assumed to be subvectors of $\vec{\theta}$; while the ξ_i need not be identical, common elements are certainly allowed. The events that create the output distortion thereby influence costs, as well, and such events can affect the output of either good individually or both goods simultaneously.

The comparative advantage emerges from this extension with some new terms:

$$\Delta_{10} = \Delta_9 - \frac{1}{2}(B_{11}-C_{11})\ \mathrm{Var}\ \phi_1 - \mathrm{Cov}\ ((\beta_1-\alpha_1);\ \phi_1)$$

$$- \frac{1}{2}(B_{22}-C_{22})\ \mathrm{Var}\ \phi_2 - \mathrm{Cov}\ ((\beta_2-\alpha_2);\ \phi_2)$$

$$- (B_{12}-C_{12})\ \mathrm{Cov}\ (\phi_1;\phi_2). \qquad (4.2.8)$$

The new terms behave just as their counterparts did in (4.2.7), and therefore require little additional interpretation. Observe, however, that since output variation under quantity control does not depend upon changes in marginal costs, there do not exist efficiency gains on the quantity side of Δ_{10}.

The value assumed by C_{12} has a marked effect on the comparative advantage. Observe initially that when $C_{12} = 0$,

$$\tilde{q}_i(\vec{\theta}) = \hat{q}_i - (\alpha_i(\vec{\theta})/C_{ii});\ i = 1,2,$$

and Δ_{10} equals Δ_8. This is certainly an expected result and is more of a check on (4.2.8) than a significant observation. When we examine the extreme values of C_{12}, however, we note that C_{12} has an effect on only output variation under price controls; indeed,

$$\lim_{C_{12}\to\infty} \tilde{q}_i(\vec{\theta}) = \lim_{C_{12}\to-\infty} \tilde{q}_i(\theta) = \hat{q}_i;\ i = 1,2. \qquad (4.2.9)$$

We can observe these limits directly from equations (4.1.6), or by reasoning geometrically as follows. The slope of an isocost curve can be determined by totally differentiating equation (4.2.2) and setting $dC = 0$; hence

$$\left.\frac{dq_2}{dq_1}\right|_{\bar{C}} = -\left[\frac{\alpha_1+C_1'+C_{12}q_2+C_{11}q_1}{\alpha_2+C_2'+C_{12}q_1+C_{22}q_2}\right]$$

Random changes in $\alpha_1(\vec{\theta})$ and $\alpha_2(\vec{\theta})$ therefore change the slope of an isocost line at any point, and, as a result, change the tangency point to a price line that is given by the optimal price orders. As $|C_{12}|$ becomes large, however, these influences become negligible, and the quantity changes implied by the slope changes disappear. We can conclude that

$$\lim_{C_{12}\to\infty} \Delta_{10} = \begin{cases} \infty\ ; & Cov(\phi_1;\phi_2) > Cov(\frac{C_{12}\alpha_2-\alpha_1C_{22}}{D};\ \frac{C_{12}\alpha_1-\alpha_2C_{11}}{D}) \\ \Delta_8; & Cov(\phi_1;\phi_2) = Cov(\frac{C_{12}\alpha_2-\alpha_1C_{22}}{D};\ \frac{C_{12}\alpha_1-\alpha_2C_{11}}{D}) \\ -\infty\ ; & Cov(\phi_1;\phi_2) < Cov(\frac{C_{12}\alpha_2-\alpha_1C_{22}}{D};\ \frac{C_{12}\alpha_1-\alpha_2C_{11}}{D}) \end{cases}$$

$$\lim_{C_{12}\to-\infty} \Delta_{10} = \begin{cases} -\infty\ ; & Cov(\phi_1;\phi_2) > Cov(\frac{C_{12}\alpha_2-\alpha_1C_{22}}{D};\ \frac{C_{12}\alpha_1-\alpha_2C_{11}}{D}) \\ \Delta_8; & Cov(\phi_1;\phi_2) = Cov(\frac{C_{12}\alpha_2-\alpha_1C_{22}}{D};\ \frac{C_{12}\alpha_1-\alpha_2C_{11}}{D}) \\ \infty\ ; & Cov(\phi_1;\phi_2) < Cov(\frac{C_{12}\alpha_2-\alpha_1C_{22}}{D};\ \frac{C_{12}\alpha_1-\alpha_2C_{11}}{D}) \end{cases}$$

The induced effect on marginal cost under quantities dominates the prices versus quantities comparison, in these cases, and the choice turns on whether the two goods vary in the correct or incorrect direction. Observe finally that (4.2.8) reduces to Δ_8 in both extremes when the output variations of both modes of control are identically correlated.

A significant change in results could be suspected when C_{ii} becomes arbitrarily large, because output variation under prices no longer disappears. Consider, for instance, good 1 and allow C_{11} to become arbitrarily large:

$$\lim_{C_{11}\to\infty} \tilde{q}_1(\vec{\theta}) = \hat{q}_1 - (\alpha_1(\vec{\theta})/C_{22}).$$

When we take this limit in the context of the comparative advantage of prices, however, we find the efficiency gain still dominating the cost side for prices, and price controls are still overwhelmingly preferred:

$$\lim_{C_{11}\to\infty} \Delta_{10} = \lim_{C_{11}\to\infty} C_{11}(Var(-\alpha_1/C_{22})+Var\ \phi_1) = \infty$$

The conclusions when B_{ii} becomes arbitrarily negative meanwhile remain exactly as described in Section 4.1, the sign of Δ_{10} turning on the sign of $(Var(\hat{q}_i(\vec{\theta})) - Var(\phi_i))$. We note, therefore, the same potentiality for a profitable policy mix as that which motivated Subsection 4.1.3. That discussion must now be repeated, since the C_{12} term in marginal costs adds a new dimension of difficulty.

4.2.3: The Profitability of Potential Mixes

We begin by comparing a mix that controls q_1 by prices and q_2 by quantities with the control of both goods by quantities. The optimal quantity orders under the second scheme remain, of course,

$$\hat{q}_{pi} = \hat{q}_i - E(\phi_i(\xi_i)); \; i = 1,2.$$

The optimal controls for the mixed scheme need to be determined. For any quantity order issued to good 2, q_{p2}, the reaction function of good 1 to any price order, p_1, is given by

$$p_1 = \alpha_1(\vec{\theta}) + C_1' + C_{11}(q_1(p_1, q_{p2} + \phi_2(\xi_2), \vec{\theta}) - \hat{q}_1)$$
$$+ C_{12}(q_{p2} + \phi_2(\xi_2) - \hat{q}_2);$$

that is

$$q_1(p_1, q_{p2} + \phi_2(\xi_2), \vec{\theta}) = \hat{q}_1 + \left(\frac{p_1 - \alpha_1 - C_1'}{C_{11}}\right) - (C_{12}/C_{11})(q_{p2} + \phi_2\ (\xi_2) - \hat{q}_2)$$

We can insert q_{p2} and $q_1(p_1, (q_{p2} + \phi_2(\xi_2)), \vec{\theta})$ into the benefit and cost functions to determine the optimal price order for good 1 and the optimal quantity order for good 2:

$$\tilde{p}_1 = C_1' = B_1', \text{ and}$$

$$\hat{q}_{p2} = \hat{q}_2 - E(\phi_2(\xi_2)),$$

so that

$$\tilde{q}_1(\theta) = \hat{q}_1 - (\alpha_1/C_{11}) - (C_{12}/C_{11})(\phi_2 - E\phi_2)$$

The optimal quantity order is thus invariant across the type of order issued to the first good; the optimal price order is similarly invariant to the control placed on the second good, but the response function is not.

Computing the comparative advantage of the mix over quantity control of both goods is an arduous task. Collecting terms on the cost side of price controls of good 1 in the same manner outlined in Subsection 4.2.1, however, we arrive at the following expression:

$$\begin{aligned}\Delta(pq/qq) = [&\tfrac{1}{2}(B_{11}+C_{11})\mathrm{Var}((-\alpha_1/C_{11})-(C_{12}/C_{11})(\phi_2-E\phi_2)) \\ &- \tfrac{1}{2}(B_{11}-C_{11})\mathrm{Var}\ \phi_1 + \mathrm{Cov}(\beta_1;((-\alpha_1/C_{11})-(C_{12}/C_{11})\phi_2-E\phi_2)) \\ &- \mathrm{Cov}((\beta_1-\alpha_1);\phi_1) + B_{12}\ \mathrm{Cov}(((-\alpha_1/C_{11})-(C_{12}/C_{11})(\phi_2-E\phi_2));\phi_2)] \\ &- (B_{12} - C_{12})\ \mathrm{Cov}\ (\phi_1;\phi_2). \qquad (4.2.10)\end{aligned}$$

The bracketed term in equation (4.2.10) is, as usual, the comparative advantage of prices over quantities for good 1, taken in the context of the induced effects on marginal costs and benefits created by quantity control of q_2. The final term represents those induced effects that would have been created were q_1 under quantity control. Since these effects are foregone by price control, their representative is subtracted from the comparative advantage of the mix.

We have thus far only repeated the analysis of Subsection 4.1.3 in a slightly more general case; nothing new has yet emerged. This situation changes, however, when we contrast the given mix with price control of both goods. The optimal orders remain the same under the mix, but the reaction functions of both goods change. The quantity response of good 1 obviously changes from

$$[\hat{q}_1 - (1/D)(C_{12}\alpha_2 - \alpha_1 C_{22})] \quad \text{to} \quad [\hat{q}_1 - E\phi_1 + \phi_1(\xi_1)]$$

The quantity response of good 2 is altered as well;

$$[\hat{q}_2 - (1/D)(C_{12}\alpha_1 - \alpha_2 C_{11})]$$

is produced when both goods are under price controls, while

$$[\hat{q}_2 - (\alpha_2/C_{22}) - (C_{12}/C_{22})(\phi_2(\xi_2) - E\phi_2)]$$

is produced under the mix. The comparative advantage of the mix is therefore not, as it has been before, simply an expression concerning good 1 and some foregone induced effects; it must also reflect the changes in the output of good 2 created by the mix. Even without recording that comparative advantage, we can conclude that the existence of one good for which quantities would be preferred does not necessarily warrant the issuance of a mixed policy in lieu of price controls on both goods.

4.2.4: Conclusions

The joint production of two goods adds a second induced effect to the list of effects that need to be considered when choosing between quantity controls and price controls. As the output of one good increases (e.g.) in response to a random influence, the marginal costs of the second good are induced to increase or decrease as C_{12} is greater than or less than zero (i.e., the goods are substitutes or complements in production). The same random event that caused the first output change may induce the output of the second good to change, as well. If this second change fights the induced change in marginal costs (e.g., an increase in output in the face of an increase in costs), the mode of control allowing such changes become less preferred. The three other permutations of the signs are similarly explained.

The profitability of a mixed policy scheme for joint products need not necessarily depend only upon changes in the output behavior of the good whose control is to be switched. The reaction function of the second good to an optimal price order depends explicitly upon that behavior through the same induced effect explained immediately above.

These results once again remain valid for any number of producers, and an increase in that number can still be shown to unambiguously favor the mode of control that creates the smaller variance in total output.

Section 4.3: The Automobile Example Extended to Nitrous Oxides

Consideration of the control of vehicular nitrous oxide emissions in the context of our previous study of carbon monoxide provides an excellent example of the simultaneous regulation of joint products. We will again illustrate the center's analysis of the prices-quantities choice by using the cost data of Dewes and the benefit data of Ahern.

Table (4.3) and Figure (4.2) summarize the cost data for nitrous oxide (NO_x) control,[5] while Table (4.4) and Figure (4.3) record the corresponding benefit data.[6] Notice that the early control devices actually caused an increase in NO_x emissions, so that the marginal cost schedule in Figure (4.2) begins with a 12% increase over the 1967 levels. Figure (4.4) then synthesizes the two marginal schedules expressed in terms of dollars per year.

The center has a choice of four possible controls in this example: price control of both pollutants, quantity control of both pollutants, and one of two possible mixes. It must, however, decide between the four before the actual cost of each of the possible control devices is known. Profit motivated variance in the emissions of both pollutants under prices will therefore still complicate the center's choice. In addition, product variability in the devices themselves exists for both pollutants under either mode of control, and must also be considered. While the center would presumably collect its own more complete information, we are forced to continue making a variety of assumptions in the course of illustrating the center's analysis with the data that we have available.

We can equip each automobile with but one control system. While system (D) overrestricts both pollutants, it seems to come the closest to being efficient with respect to both pollutants. System (C), for

[5]Dewes, op. cit., Appendix C.

[6]Ahern, op. cit., p. 198.

Table 4.3

The Cost Side for NO_x

System (A): Positive Crankcase Ventilation
System (C): 1970 Controlled Combustion System plus (A)
System (D'): Low-Lead, Low-Octane Engine plus (C)
System (D): 1971 Catalytic Exhaust Converter plus (D')

System	Emission	% Reduction	Change in Total Cost	Marginal Cost
No Control	4.0	---	---	---
(C)	7.0	-75%	.00057	
(D')	4.5	-12%	.00180	2.9×10^{-5}
(D)	2.1	48%	.00508	8.2×10^{-4}
	(gm./mi.)		($/mi.)	$/(gm./mi.)

Source: Donald Dewes, Economics and Public Policy: The Automobile Pollution Case, MIT Press, Cambridge, 1974, Appendix C.

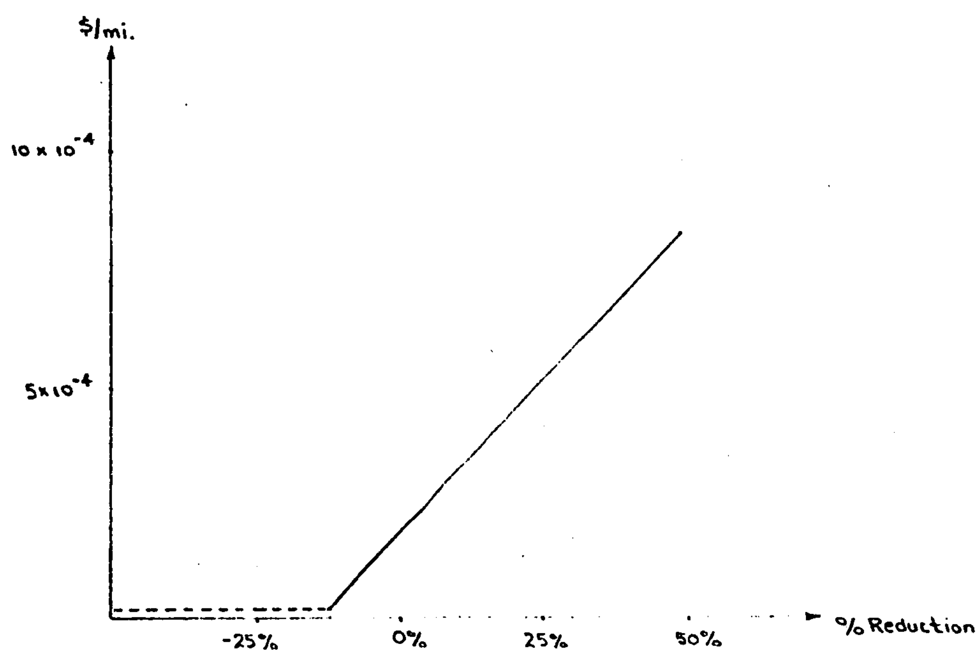

Figure 4.2: Marginal Costs for NO_x

Table 4.4

EDRA Estimates for Nitrous Oxides

1967 Levels	50% Reduction	75% Reduction
5.0×10^8	1.8×10^8	$.26 \times 10^8$

Source: William Ahern, Jr., "Measuring the Value of Emission Reductions," in Jacoby and Steinbruner, Cleaning the Air, p. 198.

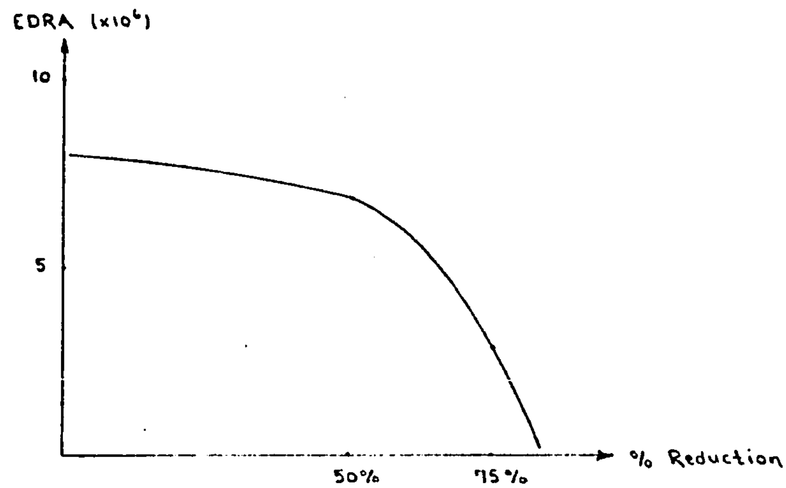

Figure 4.3: Marginal Benefits for NO_x

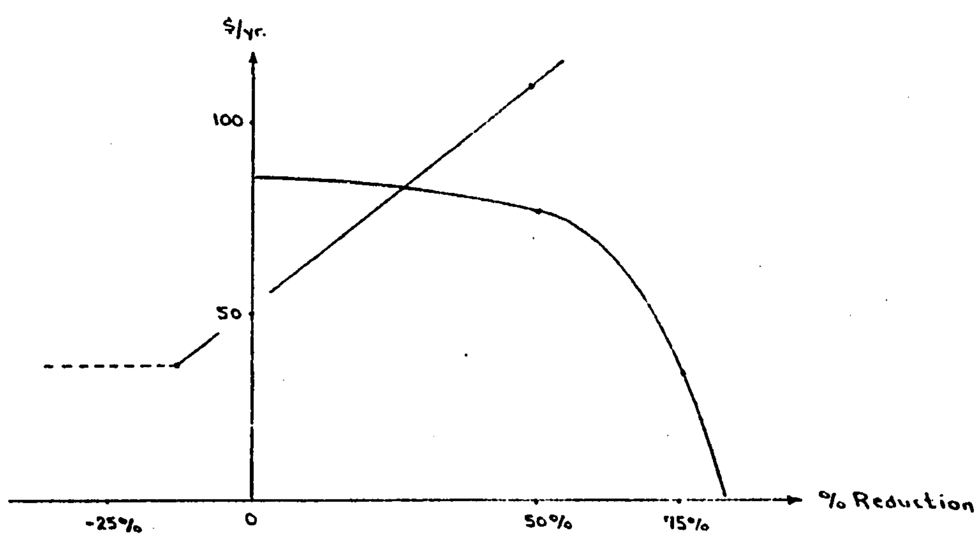

Figure 4.4

instance, actually increases emissions of nitrous oxides by 75% over the 1967 levels, even though it was deemed most efficient when carbon monoxide was considered alone. We will therefore compare prices and quantities for system (D).

Equation (4.2.8) instructs us to consider the two pollutants together. We do not, however, have information about the size of the interdependence coefficients for either costs or benefits (C_{12} and B_{12}). As a first step in our casual study, therefore, both pollutants will be considered individually and an individualized comparative advantage of prices computed for each. We will then heuristically deduce the signs of B_{12} and C_{12}, as well as the covariances in output disturbances that they multiply in the joint comparative advantage, to determine the potential effect of the cross terms on the conclusions reached in the dichotomized first step.

The analysis for carbon monoxide has been completed already for the 97% reduction achieved by system (D). Table (2.4) records the results, showing prices to be preferred regardless of the value assumed by the variance in output created by product variability under price controls (σ_p^2).

System (D) meanwhile achieves a 48% reduction in nitrous oxide emissions over the 1967 level; i.e., an average of 2.1 grams of NO_x are to be emitted per mile. The slopes of the marginal schedules in the neighborhood of 48% provide estimates for the relevant curvatures of costs and benefits. The Dewes data suggests that $C_{11}(NO_x) = \$1.8 \times 10^7$/yr. per 1%. The Ahern data, however, reveals that marginal benefits are very nearly horizontal for reductions less than 50%, so that the applicable curvature of the benefit function is very small. We can therefore presume that the cost side will dominate the comparative advantage of prices for nitrous oxides and that $B_{11}(NO_x)$ is negligible.

The marginal cost of system (D) is given by Dewes to be $\$11.5 \times 10^7$/yr. Estimates by the Environmental Protection Agency, the National Academy of Sciences, and the manufacturers suggest a standard deviation of $\$1.9 \times 10^7$ per year around the Dewes estimate.[7] Following the procedure outlined in Section 2.5, we can use this standard deviation in marginal costs to predict a profit motivated variance in NO_x emissions ($\sigma_\pi^2(NO_x)$) of .01 (gm./mi.)^2. We will use this figure as an approximation of the variance that the center would compute from its own cost data in assessing the prices-quantities comparison. We will further assume that the 20% standard deviation in emissions induced by product variability that was observed under quantity control of CO applies to NO_x as well. This assumption translates into a variance of .16 (gm./mi.)^2 under quantities at the 48% reduction level ($\sigma_q^2(NO_x)$). Sensitivity analysis will again be required to handle the corresponding variance under prices ($\sigma_p^2(NO_x)$); the three trial values will be .04, .16, and .36 (gm./mi.)^2.

We can now compute an individualized comparative advantage of prices for nitrous oxides. Table (4.4) summarizes the results for the three values of $\sigma_p^2(NO_x)$. Notice that in the first two cases, price controls are preferred, and the center should choose to regulate both pollutants with taxation schemes. When $\sigma_p^2(NO_x) = .36$ (gm./mi.)^2, however (indeed

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[7]Dewes, op. cit., p. 191.

Table 4.4

Comparative Advantage of Prices at
48% Reduction of Nitrous Oxides

$$B_{11}(NO_x) = 0$$

$$C_{11}(NO_x) = \$1.8 \times 10^7/\text{yr. per } 1\%$$

$$\sigma_\pi^2\ (NO_x) = .01\ (\text{gm./mi.})^2$$

$$\sigma_q^2\ (NO_x) = .16\ (\text{gm./mi.})^2$$

$\sigma_p^2(NO_x)$	Δ
.04	$(1/2)(1.8)(.01 + .16 - .04) \times 10^7 > 0$
.16	$(1/2)(1.8)(.01 + .16 - .16) \times 10^7 > 0$
.36	$(1/2)(1.8)(.01 + .16 - .36) \times 10^7 < 0$

when $\sigma_p^2(NO_x) > .17\ (\text{gm./mi.})^2$), the center would be well advised to mix its controls by placing nitrous oxides under quantity standards and carbon monoxide under price standards.

Equation (4.2.8) requires us to evaluate the following expression in assessing the effects of the missing cross terms of the joint benefit function on our individualized results:

$$B_{12}(\text{Cov(CO under prices; } NO_x \text{ under prices)}$$
$$- \text{Cov(CO under quantities; } NO_x \text{ under quantities))}.$$

What can we say about the signs of the components of this expression? It has been the experience of Los Angeles that carbon monoxide and nitrous oxides react to create photochemical smog when they appear together in sunlight. They are therefore more harmful when confronted together than apart, and we conclude that the missing B_{12} must be negative. If the covariance of the emissions of CO and NO_x is larger under quantities than under prices, then this entire expression is positive and price control of both pollutants receives a positive bias. The opposite result similarly holds if the covariance under prices is larger. The overall importance of this effect depends, of course, upon both the unknown absolute magnitude of B_{12} and the size of the disparity in the covariances.

The corresponding expression for the missing cross term of the joint cost function is similarly

$$C_{12}(\text{Cov}(\text{CO under prices; } NO_x \text{ under prices}) + \text{Cov}(\text{CO under quantities; } NO_x \text{ under quantities})).$$

We should expect both the product variability and the profit motivated variation to tend to move the emissions of the two pollutants in the same direction. The two covariances can therefore be presumed to be positive, so that the sign of the entire expression turns on the sign of C_{12}. Since early carbon monoxide controlling devices actually increased average nitrous oxide emissions, we can reasonably conclude that C_{12} is also positive. Hence, the missing cost term unambiguously favors price control of both pollutants.

Chapter Five

THE REGULATION OF AN INTERMEDIATE GOOD

The influence of substitutability in both consumption and production on the comparative advantage of prices was thoroughly explored in the previous chapter. We will now extend the discussion of substitution in one final direction by investigating the regulation of an intermediate good whose value is registered only through the final good that it is used to produce. Calling the final good x and the intermediate good q, we will be concerned primarily with the effect of the elasticity of substitution between q and a second factor of production, K, on the comparative advantage of prices. The existence of any profit motivated pressures on the producer of x to avoid either mode of control on q by integrating its production into his own process will also be carefully noted.

So that we can concentrate on these effects alone, we must assume that inventories of the intermediate good are maintained at a fixed level. Any fluctuation in the output of q is then reflected in deliveries and therefore registered as fluctuation in the profit-maximizing levels of the other x-producing input. Fluctuation in the output of x results. The effect of loosening this constraint on inventories, however, is easily deduced in a concluding section. Consistent with our continued specification of a cost function for q dependent only upon the output of q, we also assume that the second production factor, K, is available to the producer of x at a per unit cost of r.

Section 5.1: The Extreme Cases

The simplest way of demonstrating that the elasticity of substitution between K and q (σ) has an influence on the comparative advantage of prices is to present the two extreme cases in juxtaposition:

$$(1) \quad x = \gamma K + (1-\gamma)q, \text{ and}$$

$$(2) \quad x = \min(\gamma K;(1-\gamma)q).$$

K and q are perfect substitutes in the first example ($\sigma = \infty$); there is absolutely no substitution of inputs in the second ($\sigma = 0$). The model within which we will discuss these extremes is the analog of the simple model that has been used to initiate the discussions of each of the previous chapters. It is characterized by the following list of quite familiar assumptions:

(i) The benefit function depends entirely upon x; uncertainty is recorded by η so that $B = B(x,\eta)$.

(ii) The production of q is summarized by a cost function that depends only upon q; uncertainty here is recorded by θ and $C = C(q,\theta)$.

(iii) The random variables are jointly distributed.

(iv) The input K is available to the producer of x at the per unit cost of r.

(v) Given a quantity order, the producer of q will produce exactly that amount in the most efficient manner; that is, he will stay on his cost curve (there does not yet exist an output distortion under quantity control). Meanwhile, the producer of q will maximize profits in response to a price order with perfect knowledge of the value taken by θ; he will therefore set the price order equal to $C_1(q,\theta)$.

We further assume, for our immediate purposes, that the producer of x must use all of the input q that is produced and that he is charged the expected marginal value product of q in case (1) and some non-zero price in case (2).[1] The producer of x is finally presumed to be public spirited (or government controlled, e.g.) in the sense that in deciding the K-response to the amount of q that he receives, he maximizes expected social benefits minus the costs that he incurs. These final two assumptions appear to be quite restrictive; in subsequent sections, however, our results will be shown to be significantly independent of these behavioral characterizations.

We consider the case of perfect substitutes first, and observe that the x-producing firm (x-firm) will react to an input delivery of q by solving

$$E[B_1((\gamma K(q) + (1-\gamma)q),\eta)\cdot\gamma] = r \qquad (5.1.1)$$

for K(q). Equation (5.1.1) therefore implicitly defines the K response to any quantity q. The center can then determine the optimal quantity order, $\hat{q}_p$, by taking K(q) and solving

[1] All that is required for our purposes is that production occur on the corners of the right angle isoquants implied by case (2). Were we to worry about the profitability of producing x, a non-zero upper bound for this price would also exist.

$$\max_q \, [E[B((\gamma K(q) + (1-\gamma)q),\eta) - rK(q) - C(q,\theta)]].$$

The optimal price order, $\tilde{p}$, is similarly determined by noting that the reaction function of the q-producing firm (q-firm) to any price order,

$$p = C_1(q(p,\theta),\theta),$$

can be summarized by

$$q(p,\theta) \equiv h(p,\theta).$$

The center must therefore solve the following

$$\max_p \, \{E[B(\gamma K(h(p,\theta)) + (1-\gamma)(h(p,\theta)) - rK(h(p,\theta)) - C(h(p,\theta),\theta)]\}.$$

The amount of q delivered to the x-firm under price control is clearly $h(\tilde{p},\theta)$. As a result, the amount of x produced under quantity control is characterized by

$$E[B_1((\gamma K(\hat{q}_p) + (1-\gamma)\hat{q}_p),\eta) \cdot \gamma] = r,$$

while the amount of x produced under price control for any θ is similarly given by

$$E[B_1((\gamma K(h(\tilde{p},\theta)) + (1-\gamma)h(\tilde{p},\theta)),\eta) \cdot \gamma] = r.$$

We can conclude, since the benefit function is arbitrary, that for any θ,

$$[\gamma K(\hat{q}_p) + (1-\gamma)\hat{q}_p] = [\gamma K(h(\tilde{p},\theta)) + (1-\gamma)h(\tilde{p},\theta)];$$

that is, as the states of nature on the cost side change, the amount of q delivered to the x-firm changes, but K is adjusted so that the output of x remains constant. In particular, the output of x, and thus the quantity inserted into the benefit function, under price control of q by $\tilde{p}$ is always equal to the output of x produced under quantity control of q by $\hat{q}_p$. The comparative advantage of prices, being a relative measure, therefore reflects only the cost effects of output variation. Since the efficiency gain under prices has been shown to always dominate the increase in expected costs created by output variation, prices are preferred unambiguously.

The second case specifies that absolutely no substitution be allowed between K and q in the production of x. Since we have established a price for q that guarantees that x is produced on the corner of the right angle isoquants, we can observe immediately that $x = (1-\gamma)q$. The benefit function is therefore easily written as a function of q, and it becomes a simple matter to compute either an optimal quantity order or an optimal price order for q. The output of good x, however, now varies as the production of q varies, so that the comparative advantage of prices captures a loss in expected benefits under prices as well as the cost effect just noted; its sign is thus in doubt.

These few observations uncover a fundamental difference in the price-quantity comparison that is created by shifting the elasticity of substitution from one extreme to the other. The remainder of this chapter is devoted to putting this difference into perspective by considering the intermediate examples. The precise formulations of the two cases with which we have just motivated the question are contained therein.

Section 5.2: Output Responses and Expected Profits

The cases of intermediate substitutability of inputs are all represented by the constant elasticity of substitution production function

$$x = (\gamma K^{\rho} + (1-\gamma)q^{\rho})^{1/\rho} \tag{5.2.1}$$

as ρ ranges from $-\infty$ to 1. Equation (5.2.1) reduces to the Cobb-Douglass form

$$x = K^{\gamma}\, q^{(1-\gamma)} \tag{5.2.2}$$

when ρ approaches zero;[2] this is a special case whose importance will become clear. For the moment, however, we select an arbitrary representative of this class and deduce the optimal response of x to varying sizes of the deliveries of input q. Assumptions (i) through (v) of Section 5.1 characterize the model "below" the final good; a variety of behavioral assumptions for the x-firm and pricing assumptions for the intermediate good q will be explored. The profitability of producing x will also be explored as a peripheral interest in each case, under the assumption that x is sold at a price equal to the marginal benefits that actually occur (i.e., $B_1(x,\eta)$). The planner must always know when his controls on q create pressures on the x-firm to try to avoid those controls by producing q internally.

5.2.1: A First Model

We begin our analysis with the model in which perfect substitutes

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[2] Henderson and Quandt, Microeconomic Theory, pp. 87-88.

were studied above. The x-firm selects his K response to the delivery of q by maximizing expected benefits minus the costs that he incurs. He is charged the expected marginal value product for each unit of q that is delivered. The analysis of this case will be constructed so that we will be able to draw heavily upon these present results in discussing subsequent models.

The optimal level of production for the intermediate good, $\hat{q}_o$, and the optimal K response, $\hat{K}_o$, can be determined by solving

$$\max_{q;K} [E(B((\gamma K^\rho + (1-\gamma)q^\rho)^{1/\rho},\eta) - rK - C(q,\theta))],$$

for the given benefit and cost functions. The first order conditions that characterize these optima require that

$$E(B_1 \cdot x_K) = r; \qquad (5.2.3a)$$

$$E(B_1 \cdot x_q) = E(C_1(\hat{q}_o,\theta)). \qquad (5.2.3b)$$

The shapes of the functions guarantee the existence of $\hat{q}_o$ and $\hat{K}_o$, so that we can expand the benefit function around $x_o \equiv (\gamma\hat{K}_o^\rho+(1-\gamma)\hat{q}_o^\rho)^{1/\rho}$, and the cost function around $\hat{q}_o$. Consistent with our previous models, we assume that these approximations are of the following form:

$$B(x,\eta) = b(\eta) + (B'+\beta(\eta))(x-x_o) + \frac{1}{2} B_{11}(x-x_o)^2; \qquad (5.2.4a)$$

$$C(q,\theta) = a(\theta) + (C'+\alpha(\theta))(q-\hat{q}_o) + \frac{1}{2} C_{11}(q-\hat{q}_o)^2. \qquad (5.2.4b)$$

We can now rewrite equations (5.2.3) for future reference:

$$E[(B'+\beta(\eta))(\frac{(\gamma\hat{K}_o^\rho + (1-\gamma)\hat{q}_o^\rho)^{1/\rho}}{\hat{K}_o})^{1-\rho}] = r; \quad (5.2.3a)'$$

$$E[(B'+\beta(\eta))(\frac{(\gamma\hat{K}_o^\rho + (1-\gamma)\hat{q}_o^\rho)^{1/\rho}}{\hat{q}_o})^{1-\rho}] = C'. \quad (5.2.3b)'$$

One simple observation should be made in passing; equations (5.2.3)' can be combined to reveal that

$$(\hat{K}_o/\hat{q}_o)^{1-\rho} = (\gamma C'/(1-\gamma)r);$$

as a result,

$$\hat{K}_o = (\gamma C'/(1-\gamma)r)^\sigma \hat{q}_o \equiv (\bar{C}'/\bar{r})^\sigma \hat{q}_o.^3 \quad (5.2.5)$$

The output x_o can therefore be written

$$x_o = \hat{q}_o\ (\gamma(\bar{C}'/\bar{r})^{\sigma\rho} + (1-\gamma))^{1/\rho} \equiv \hat{q}_o(A(\rho)).$$

In terms of this more compact notation, equations (5.2.3) reduce to

$$B'(A(\rho))^{1-\rho} = C'; \quad (5.2.3b)''$$

$$B'(A(\rho)/\bar{C}'/\bar{r})^\sigma)^{1-\rho} = r. \quad (5.2.3a)''$$

Were the center to impose quantity regulation on the intermediate good, the optimal quantity order would clearly be $\hat{q}_o$. The x-firm would determine its optimal K response, $\hat{K}$, upon delivery of $\hat{q}_o$, by solving

[3]Under our specification of the CES production function, $\sigma = 1/(1-\rho)$.

$$E[(B'+\beta(\eta)+B_{11}((\gamma\hat{K}^\rho+(1-\gamma)\hat{q}_o^\rho)^{1/\rho}-x_o))\cdot(\frac{(\gamma\hat{K}^\rho+(1-\gamma)\hat{q}_o^\rho)^{1/\rho}}{\hat{K}})^{1-\rho}]=r, \quad (5.2.6)$$

the first order condition of its decision rule. Equation (5.2.3a) asserts that $\hat{K}_o$ is the solution of (5.2.6), and we can conclude that x_o is indeed produced when a quantity control of $\hat{q}_o$ is placed on the production of q. Observe that (5.2.3b) then dictates that a per unit charge for q of C' is levied upon the x-firm as it accepts $\hat{q}_o$.

Any CES production function is linearly homogeneous, so that Euler's theorem implies that

$$\hat{q}_oA(\rho) = (\frac{\hat{q}_oA(\rho)}{\hat{q}_o(\bar{C}'/\bar{r})^\sigma})^{1-\rho}\hat{q}_o(\bar{C}'/\bar{r})^\sigma + (\frac{\hat{q}_oA(\rho)}{\hat{q}_o})^{1-\rho}\hat{q}_o. \quad (5.2.7)$$

We can investigate the expected profits of the x-firm under optimal quantity control of q by multiplying both sides of (5.2.7) by $(B'+\beta(\eta))$ and taking expected values:

$$\text{(Expected Revenues)} = B'(\hat{q}_oA(\rho))$$

$$= B'(\frac{A(\rho)}{(\bar{C}'/\bar{r})^\sigma})^{1-\rho}\hat{q}_o(\bar{C}'/\bar{r})^\sigma + B'(\frac{A(\rho)}{(\bar{C}'/\bar{r})^\sigma})^{1-\rho}\hat{q}_o$$

$$= r(\bar{C}'/\bar{r})^\sigma\hat{q}_o + C'\hat{q}_o.$$

The term $(r(\bar{C}'/\bar{r})^\sigma\hat{q}_o)$ is the total expenditures on K in producing x_o; $C'\hat{q}_o$, total expenditure on q. Expected economic profits are therefore zero, and there does not exist pressure to avoid the quantity control on q by integrating the production of q into the production of x.

The computation of an optimal price order for q, designated p, is more involved. The standard point of departure is the price reaction function of the q-firm:

$$h(p,\theta) = \hat{q}_o + (\frac{p-\alpha-C'}{C_{11}}),$$

for any price order p. The K response to $h(p,\theta)$, designated by $K(h(p,\theta))$, is determined by the first order condition of the x-firm with respect to K:

$$E[(B'+\beta(\eta)+B_{11}(\frac{p-\alpha-C'}{C_{11}}))\left[\frac{(\gamma(K(h(p,\theta))^{\rho}+(1-\gamma)(\hat{q}_o+(\frac{p-\alpha-C'}{C_{11}}))^{\rho})^{1/\rho}}{K(h(p,\theta))}\right]^{1-\rho}] = r.$$

(5.2.8)

Equation (5.2.8) reduces to

$$(B' + B_{11}(\frac{p-C'}{C_{11}}))(\frac{A(\rho)}{(\bar{C}'/\bar{r})^{\sigma}})^{1-\rho}) = r \qquad (5.2.8)'$$

if we assert that

$$K(h(p,\theta)) = (\bar{C}'/\bar{r})^{\sigma}h(p,\theta)$$

$$= \hat{K}_o + (\bar{C}'/\bar{r})^{\sigma}(\frac{p-\alpha-C'}{C_{11}}). \qquad (5.2.9)$$

The center meanwhile determines the optimal price order by solving

$$\max_p [E(B((\gamma(K(h(p,\theta)))^{\rho} + (1-\gamma)(h(p,\theta))^{\rho})^{1/\rho},\eta) - rK(h(p,\theta)) - C(h(p,\theta),\theta)],$$

the first order condition for which is

$$E[(B'+\beta(\eta)+B_{11}(\frac{\tilde{p}-\alpha-C'}{C_{11}}))((\frac{(\gamma(K(\tilde{p},\theta))^{\rho}+(1-\gamma)(\hat{q}_o+(\frac{\tilde{p}-\alpha-C'}{C_{11}}))^{\rho})^{1/\rho}}{K(\tilde{p},\theta)})^{1-\rho}\frac{\partial K}{\partial q}\frac{\partial h}{\partial p}$$

$$+ (\frac{\gamma(K(\tilde{p},\theta))^{\rho}+(1-\gamma)(\hat{q}_o+(\frac{\tilde{p}-\alpha-C'}{C_{11}}))^{\rho})^{1/\rho}}{(\hat{q}_o + (\frac{\tilde{p}-\alpha-C'}{C_{11}}))})^{1-\rho}\frac{\partial h}{\partial p}) - r\frac{\partial K}{\partial q}\frac{\partial h}{\partial p} - \tilde{p}(\frac{\partial h}{\partial p})] = 0$$

(5.2.10)

Observe that equation (5.2.8) is embedded in (5.2.10) with a multiplicative factor of $(\frac{\partial K}{\partial q}\frac{\partial h}{\partial p})$. That factor is only $(1/C_{11})$ and (5.2.10) reduces to

$$E[(B'+\beta(\eta)+B_{11}(\frac{\tilde{p}-\alpha-C'}{C_{11}}))(\frac{\gamma(K(\tilde{p},\theta))^{\rho}+(1-\gamma)(\hat{q}_o+(\frac{\tilde{p}-\alpha-C'}{C_{11}}))^{\rho})^{1/\rho}}{(\hat{q}_o+(\frac{\tilde{p}-\alpha-C'}{C_{11}}))})^{1-\rho}(C_{11})^{-1}]$$

$$= \tilde{p}(C_{11})^{-1} \qquad (5.2.11)$$

The left hand side of (5.2.11) is the expected value product of q. Even without solving (5.2.11), we can therefore note that the center is optimally buying and selling q at the same price. The assertion that $\tilde{p} = C'$ requires that (5.2.11) be reduced to

$$B'(A(\rho))^{1-\rho} = C' \qquad (5.2.11)'$$

and that (5.2.8)' be reduced to

$$B'(\frac{A(\rho)}{(\bar{C}'/\bar{r})})^{1-\rho} = r$$

Equations (5.2.3) verify the validity of both of these final forms, and we can conclude that our assertions were correct:

$$\tilde{p} = C'$$

$$\tilde{q}(\theta) = \hat{q}_o - (\alpha(\theta)/C_{11}), \text{ and}$$

$$K(\theta) = \hat{K}_o = (\bar{C}'/\bar{r})^{\sigma}(\alpha(\theta)/C_{11}).$$

The profitability of the x-firm under price control of q creates a special problem for the center. Euler's theorem can be employed a second time to evaluate the expected economic profits of the x-firm:

$$(\hat{q}_o-\alpha(\theta)/C_{11})A(\rho) = (A(\rho)/(\bar{C}'/\bar{r})^{\sigma})^{1-\rho}(\bar{C}'/\bar{r})^{\sigma}(\hat{q}_o-\alpha(\theta)/C_{11})$$

$$+ (A(\rho))^{1-\rho}(\hat{q}_o-\alpha(\theta)/C_{11}).$$

Multiplying both sides by $(B' + \beta(\eta) + B_{11}(\bar{C}'/\bar{r})^{\sigma}(-\alpha(\theta)/C_{11}))$ and taking expected values, we see that

$$\text{(Expected Revenues)} = [B'(A(\rho)/(C'/r)^{\sigma})^{1-\rho}(\bar{C}'/\bar{r})^{\sigma}(\hat{q}_o)]$$

$$+ [B'(A(\rho))^{1-\rho}(\hat{q}_o)]$$

$$+ [(A(\rho)/(C'/r)^{\sigma})^{1-\rho}\mathrm{Cov}((\beta+B_{11}(-\alpha/C_{11});(\bar{C}'/\bar{r})^{\sigma}(\hat{q}_o-\alpha/C_{11}))$$

$$+ (A(\rho))^{1-\rho}\mathrm{Cov}((\beta+B_{11}(-\alpha/C_{11});(\hat{q}_o-\alpha/C_{11}))]. \qquad (5.2.12)$$

The first term in (5.2.12) represents the expected total expenditure for K; the second term, the expected total cost of q. The sum of these expressions is consequently the total expected costs of running the x-firm. Expected profits are therefore crucially dependent upon the final term of (5.2.12), the covariance of shifts in the marginal benefit function and shifts in the output of x.[4] Output tends to increase as the selling price for x increases if that covariance is positive, even though per unit factor costs remain constant. Expected revenues exceed expected costs in this case, and the firm averages a positive profit. The opposite situation occurs, however, when the covariance is negative; there would then

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[4]Algebraically combining the arguments of the final bracketed term of equation (5.2.12) by Euler's theorem as stated above, we can note that it is equal to $(\mathrm{Cov}((\beta(\eta) + B_{11}(-\alpha/C_{11}));x(\theta))$.

exist pressure on the x-firm to avoid this loss situation by manufacturing its own q, regulated internally by quantities. We have already shown that such integration would be economically viable, yielding zero expected economic profits.

Careful examination of the crucial covariance reveals a permanently negative subterm:

$$\mathrm{Cov}(B_{11}((\bar{C}'/\bar{r})(-\alpha/C_{11}));((\bar{C}'/\bar{r})(-\alpha/C_{11}))) = B_{11}\mathrm{Var}((\bar{C}'/\bar{r})(-\alpha/C_{11})) < 0$$

This term represents the loss caused by the correlation of changes in output and their induced changes in marginal benefits; recall that the actual level of marginal benefits has been assumed to be the selling price for x. A significant positive correlation between output changes and uninduced changes in marginal benefits would therefore be required to create positive expected profits. The mere independence of η and θ could create one of many possible circumstances in which prices could be preferred, but for which the expected economic profits of the x-firm could be negative. The profitability of the production of x can therefore be a serious problem that warrants further investigation.

5.2.2: Variation in the Pricing Policy and the Profitability of x

The fundamental assumption of the preceding analysis has been that the x-firm is required to use all of the q that is delivered to him. Observe that the producer of x is therefore able to compute his K response to q independent of the price that he is charged for q; he simply takes the delivery of q as given and maximizes the expected value of his objective function with respect to K. In this, as well as in the previous subsection, that objective function is social benefits; it will be expected profits in the next subsection. Since the K-response is invariant to our pricing policy on q in either case, we are not required to rework the preceding analysis to engineer any potential pricing changes. Only the profitability of the x-firm will reflect the change.

Perhaps the simplest procedure would be to deliver the q produced under either prices or quantities to the x-firm gratus. The above argument implies that the optimal quantity order remains $\hat{q}_o$, the optimal price order remains C', and the x response to θ under such price control remains

$$x_o - (\alpha/C_{11})\ A(\rho).$$

There is, however, an average transfer of revenue in the amount of $C'\hat{q}_o$ under either prices or quantities from the center to the x-firm.[5] One would expect that this transfer would render the x-firm profitable, even given the covariance difficulties listed above, and thus eliminate the pressure to integrate production.

A policy with less severe distributional effects is suggested by the following example. Suppose that the center were willing to postpone payment for q until after the corresponding x had been produced and its actual marginal value known. The center could then charge the x-firm a per unit fee for q precisely equal to the actual marginal value product of q in producing x. All of the optimal orders and output responses

[5] For any θ, the payment foregone under the gratus scheme is $C'(\hat{q}_o-(\bar{C}'/\bar{r})^{\sigma}(\alpha(\theta)/C_{11})))$; the expected value of this transfer is clearly $C'\hat{q}_o$.

then remain the same, but expected revenues of the x-firm would differ from expected costs by only

$$\mathrm{Cov}(\beta + B_{11}\ ((\bar{C}'/\bar{r})^{\sigma}(-\alpha/C_{11}));\ (\frac{A(\rho)}{\bar{C}'/\bar{r})^{\sigma}})^{1-\rho}\ (-\alpha/C_{11})).$$

The troublesome covariance term has been essentially "cut in half," reflecting the charging of a nonstochastic price for K.

Reviewing this second variation makes it clear that the difficulty in the profitability of the x-firm under price control lies in the fact that

(expected marginal value product) • (quantity of input) ≠
expected((marginal value product) • (quantity of input)).

Were the center to charge the actual marginal value product of K for each unit of K used in producing x, the production of x would net an expected economic profit of zero under prices as well as under quantities. If, in addition, the center were to require that the expected value of the actual marginal value product of K be precisely equal to r, the first order condition of the x-firm with respect to K would remain precisely the same and the K response to q would also be preserved.[6]

The preceding paragraphs have recorded a pricing policy that both solves the profitability question, thereby alleviating the pressure to integrate, and leaves the analysis of Section 5.1 unaltered. It will, however, greatly facilitate the exposition of further complications in the model if we continue to charge the x-firm a per unit charge of the

[6]This second condition guarantees, in addition, that the expected revenues of the center that is now buying K at r and selling K at the actual marginal value product of K in producing x are zero.

expected marginal value product for q and r for K. We will consequently maintain that presumption throughout the remainder of this chapter. The reader should always be aware that the following analysis is equally valid under the above pricing schemes that guarantee profitability.

5.2.3: Profit Maximization by the Producer of the Final Good

The producer of the final good, x, has been assumed thus far to be a maximizer of social benefits. We will presently extend the analysis to model a profit maximizing producer who is equipped with only a distribution of the price at which he can sell his product. We assume, in particular, that the x-producer believes that the price of x varies with η (i.e., $p_x = p_x(\eta)$), and that η is distributed by $g(\eta)$. We further presume, for the moment, that our producer knows the correct mean of prices:

$$E_g(p_x(\eta)) \equiv \int_{\eta} p_x(\eta)\ g(\eta)\ d\eta = B' \qquad (5.2.13)$$

The output responses under these conditions will now be shown to be precisely those responses that were observed for the socially motivated producer of Section 5.1.

Were the center to issue a quantity order of $\hat{q}_o$ to the q-firm, the profit maximizing K-response of the x-firm would be given by the first order condition that

$$E_g[(p_x(\eta))(\frac{(\gamma(K(\hat{q}_o))^{\rho}+(1-\gamma)\hat{q}_o^{\rho}\ )^{1/\rho}}{K(\hat{q}_o)})^{1-\rho}] = r. \qquad (5.2.14)$$

Combining equations (5.2.13) and (5.2.14), we note that

$$B'\left(\frac{(\gamma(K(\hat{q}_o))^{\rho}+(1-\gamma)\hat{q}_o^{\rho})^{1/\rho}}{K(\hat{q}_o)}\right)^{1-\rho} = r.$$

Equation (5.2.3)' then guarantees that

$$K(\hat{q}_o) = \hat{K}_o = (\bar{C}'/\bar{r})^{\sigma}\ \hat{q}_o;$$

the profit maximizing response to $\hat{q}_o$ remains intact, and

$$E_g[(p_x(\eta)(A(\rho)/(\bar{C}'/\bar{r})^{\sigma})^{1-\rho}] = r. \qquad (5.2.14)'$$

Equation (5.2.14)' can be used to determine the K-response under optimal price control of q. For any given price order, the response of the q-firm is again summarized by

$$h(p,\theta) = \hat{q}_o + \left(\frac{p-\alpha-C'}{C_{11}}\right).$$

The K-response of the x-firm is then defined by

$$E_g[(p_x(\eta))\left(\frac{(\gamma(K(h(p,\theta)))^{\rho}+(1-\gamma)(h(p,\theta))^{\rho})^{1/\rho}}{K(h(p,\theta))}\right)^{1-\rho}] = r. \qquad (5.2.15)$$

Equation (5.2.15) reduces to (5.2.14)' only if

$$K(h(p,\theta)) = (\bar{C}'/\bar{r})^{\sigma}\ h(p,\theta)$$

and the optimal price order, $\tilde{p}$, is again defined by equation (5.2.11). The optimal price order remains C', as a result, and

$$\tilde{q}(\theta) = \hat{q}_o - \alpha(\theta)/C_{11};$$
$$\tilde{K}(\theta) = \hat{K}_o - (\bar{C}'/\bar{r})^{\sigma}(\alpha(\theta)/C_{11}).$$

The profit maximizing responses are thus precisely the socially optimal responses derived in the previous section.

The profitability problem that exists under the current input pricing scheme inherits a second dimension in this case: the subjective expected profitability of the x-firm as viewed by the producer of x, himself. We can infer from equation (5.2.12) that expected profits under price control of q depend crucially on the subjective covariance of $p_x(\eta)$ and $x(\theta) = (\hat{q}_o-(\alpha(\theta)/C_{11}))(A(\rho)))$. Were the x-producer to feel that this covariance is negative, he would expect subzero profits and experience pressure to avoid such control of q. It should be clear, however, that the input pricing scheme outlined at the end of the last subsection will solve not only the actual profitability problem, but also this subjective profitability problem.

The assumption that the x-firm possesses the correct price mean is not as restrictive as it might initially seem. There do exist policy options for the center that will neutralize the effects of an incorrect mean that lie well short of providing the correct distribution in its entirety. Suppose, for example, that

$$E_g(p_x(\eta)) = P_x \neq B'.$$

Were the center to assert that the per unit cost of K would be

$$[r - (B'-P_x)(A(\rho)/(\bar{C}'/\bar{r})^{\sigma})^{1-\rho}] \equiv A,$$

rather than simply r, the first order condition with which the x-firm determines its K-response to $\hat{q}_o$ would become

$$E_g[(p_x(\eta))(\frac{(\gamma(K(\hat{q}_o))^\rho+(1-\gamma)(\hat{q}_o)^\rho)^{1/\rho}}{(\bar{C}'/\bar{r})^\sigma})^{1-\rho}] = A. \quad (5.2.16)$$

Inserting $K(\hat{q}_o) = (\bar{C}'/\bar{r})^\sigma \hat{q}_o$ into (5.2.16) confirms that the socially optimal K-response has been preserved. A similar argument extends this conclusion to price controls, as well.

Having demonstrated a policy that will neutralize the effects of an incorrect price mean when the center knows of the error, we close this subsection by asking what such knowledge has allowed the center to avoid. We, therefore, assume that the center supposes that the x-producer possesses the correct mean when, in fact, he does not. The optimal quantity order has been shown to be $\hat{q}_o$, but the K-response to $\hat{q}_o$ is determined by solving the analog to equation (5.2.14):

$$p_x(\frac{(\gamma(K_\varepsilon(\hat{q}_o))^\rho+(1-\rho)(\hat{q}_o)^\rho)^{1/\rho}}{K_\varepsilon(\hat{q}_o)})^{1-\rho} = r. \quad (5.2.14)''$$

There exists a positive real number ε such that

$$K_\varepsilon(\hat{q}_o) = \varepsilon\hat{K}_o = \varepsilon(\bar{C}'/\bar{r})^\sigma \hat{q}_o;$$

we thus know from (5.2.14)" that

$$p_x(\frac{(\gamma(\varepsilon(\bar{C}'/\bar{r}))^\rho+(1-\gamma))^{1/\rho}}{\varepsilon(\bar{C}'/\bar{r})^\sigma})^{1-\rho} = r. \quad (5.2.17)$$

The optimal price order is meanwhile $\hat{p} = C'$, and the K-response to $\tilde{q}(\theta)$ is given by

$$p_x(\frac{(\gamma\tilde{K}_\varepsilon(\tilde{q}(\theta))^\rho+(1-\gamma)(\tilde{q}(\theta))^\rho)^{1/\rho}}{\tilde{K}_\varepsilon(\tilde{q}(\theta)})^{1-\rho} = r. \quad (5.2.18)$$

Were $\tilde{K}_\varepsilon(\theta) = \varepsilon(\bar{C}'/\bar{r})^\sigma \bar{q}(\theta)$, equation (5.2.18) would reduce to equation (5.2.17) and equality would be assured. We conclude that

$$\tilde{K}_\varepsilon(\theta) = \varepsilon\hat{K}_o - \varepsilon(\bar{C}'/\bar{r})^\sigma(\alpha(\theta)/C_{11}) = \varepsilon\tilde{K}(\theta).$$

The production of x under either mode of control is, as should be expected, suboptimal. The determination of the relative merits of prices and quantities in this condition remains, nonetheless, a significant question.

5.2.4: The Output Distortion

We have thus far ignored the potential inability of the producer of q to fulfill a quantity order exactly. To correct this omission, we will now introduce the familiar output distortion into a model in which the producer of x maximizes social benefits and is charged the expected marginal value product for q. The quantity delivered to the x-firm, q_d, is assumed to be additively related to the quantity ordered, q_p, as follows:

$$q_d = q_p + \phi(\xi).$$

The cost function must also reflect the addition of this distortion and is represented in approximation by:

$$C(q,\theta,\xi) = a(\theta,\xi) + (C'+\alpha(\theta,\xi))(q-\hat{q}_o) + \frac{1}{2}C_{11}(q-\hat{q}_o)^2.$$

We need worry only about a change in the optimal quantity order, since $\phi(\xi)$ effects only the quantity mode. The optimal price order for q remains C', while the output response of the x-firm is still

$$\hat{x}(\theta,\xi) = (\hat{q}_o - \alpha(\theta,\xi)/C_{11})\ A(\rho).$$

The center will determine the optimal quantity order, $\hat{q}_p$, by maximizing expected benefits minus expected costs. The first order condition that it therefore confronts is:

$$E[((B'+\beta(\eta)+B_{11}(\hat{q}_d(\xi)-\hat{q}_o))((\frac{(\gamma(K(\hat{q}_d(\xi)))^{\rho}+(1-\gamma)(\hat{q}_d(\xi))^{\rho})^{1/\rho}}{K(\hat{q}_d(\xi))})^{1-\rho}\frac{\partial K}{\partial q}$$

$$+ (\frac{(\gamma(K(\hat{q}_d(\xi)))^{\rho}+(1-\gamma)(\hat{q}_d(\xi))^{\rho})^{1/\rho}}{\hat{q}_d(\xi)})^{1-\rho})] - r\frac{\partial K}{\partial q}$$

$$- (C' + \alpha + C_{11}\ (\hat{q}_p + \phi(\xi) - \hat{q}_o)) = 0.$$

The x-firm will meanwhile select its K-response to a delivery of $\hat{q}_p+\phi(\xi)$ by solving

$$E[(B'+\beta(\eta) + B_{11}((\gamma(K(\hat{q}_p+\phi(\xi)))^{\rho} + (1-\gamma)(\hat{q}_p+\phi(\xi))^{\rho})^{1/\rho} - \hat{q}_o\ A(\rho))]$$

$$\times [\frac{\gamma(K(\hat{q}_p+\phi(\xi)))^{\rho}+(1-\gamma)(\hat{q}_p+\phi(\xi))^{\rho})^{1/\rho}}{K(\hat{q}_p+\phi(\xi))}]^{1-\rho} = r. \qquad (5.2.20)$$

Were we to assert that $\hat{q}_p = \hat{q}_o - E\phi(\xi)$ and

$$K(\hat{q}_p+\phi(\xi)) = (\bar{C}'/\bar{r})^{\sigma}(\hat{q}_p+\phi(\xi)),$$

then equations (5.2.19) and (5.2.20) would reduce to

$$B'(A(\rho))^{1-\rho} = C', \text{ and} \qquad (5.2.19)'$$

$$B'(A(\rho)/\bar{C}'/\bar{r})^{\sigma})^{1-\rho} = r. \qquad (5.2.20)'$$

The validity of these last two equations is guaranteed by equations

(5.2.3)', and we have found the optimal quantity order:

$$\hat{q}_p = \hat{q}_o - E\phi(\xi), \quad \text{and} \quad \hat{q}_d = \hat{q}_o - E\phi + \phi(\xi).$$

The reader should note that this analysis is valid under any of the pricing models presented in the preceding two subsections. The production of x will be

$$(\hat{q}_o - E\phi + \phi(\xi))\, A(\rho),$$

in all cases, save the one involving an uncorrected subjective price distribution error made by a profit maximizing producer of x. Output will be

$$(\hat{q}_o - E\phi + \phi(\xi))(\gamma(\varepsilon(\bar{C}'/\bar{r})^{\sigma})^{\rho} + (1-\gamma))^{1/\rho}$$

in this lone exception.

Section 5.3: The Comparative Advantage of Prices

We will explore the comparative advantage of prices in the context of the output distortion that we have just reintroduced. Identical quantity responses have been demonstrated in all but one of the various behavioral and price setting combinations presented in Section 5.2. Postponing the exception until the end of this section (the profit maximizing production of x under an incorrect subjective price mean), we can assert that the optimal quantity order is $\hat{q}_p = \hat{q}_o - E(\phi(\xi))$. Actual deliveries of q are then

$$\hat{q}_d(\xi) = \hat{q}_p + \phi(\xi),$$

and the output of the final good is specified by

$$\hat{x}(\xi) = A(\rho)(\hat{q}_d(\xi)). \tag{5.3.1a}$$

The optimal price order is meanwhile C', so that

$$\tilde{q}(\theta,\xi) = \hat{q}_o - (\alpha(\theta,\xi)/C_{11}), \text{ and}$$

$$\tilde{x}(\theta,\xi) = (\tilde{q}(\theta,\xi))\ A(\rho). \tag{5.3.1b}$$

The comparative advantage of prices is now computable:

$$\Delta(\rho) = \frac{1}{2} B_{11}\ (\mathrm{Var}(\tilde{x}(\theta,\xi)) - \mathrm{Var}(\hat{x}(\xi))) + \mathrm{Cov}\ (\beta(\eta);(\hat{x}-\tilde{x}))$$

$$+ \frac{1}{2} C_{11}\ (\mathrm{Var}(\tilde{q}(\theta,\xi)) + \mathrm{Var}(\hat{q}_d(\xi))) - \mathrm{Cov}(\alpha(\theta,\xi);\hat{q}_d(\xi)). \tag{5.3.2}$$

The terms on the cost side of (5.3.2) are all familiar from Chapter Two. The benefit side is equally familiar when we recall that only the final good is registered in the benefit function. Variation in the output of q must be translated into variation in the output of x before its effect on the level of expected benefits is recorded. The benefit side of (5.3.2) therefore compares the losses or gains in expected benefits created by this induced variation in x under both modes of control, in the context of a randomly shifting marginal benefit schedule. The more detailed interpretation found in the second chapter is thus perfectly applicable.

We can express equation (5.3.2) entirely in terms of q by employing equations (5.3.1):

$$\Delta(\rho) = \frac{1}{2} B_{11} (A(\rho))^2 (\mathrm{Var}(\tilde{q}(\theta,\xi)) - \mathrm{Var}(\hat{q}_d(\xi)))$$

$$+ A(\rho) \mathrm{Cov}(\beta(\eta);\hat{q}_d(\xi)-\tilde{q}(\theta,\xi)) - \mathrm{Cov}(\alpha(\theta,\xi);\hat{q}_d(\xi))$$

$$+ \frac{1}{2} C_{11} (\mathrm{Var}(\tilde{q}(\theta,\xi)) + \mathrm{Var}(\hat{q}_d(\xi))). \qquad (5.3.2)'$$

The elasticity of substitution between K and q appears only in the multiplicative factor $A(\rho)$; it is the same factor that translates quantities of the intermediate good into quantities of the final good. Equation (5.3.2)' strongly suggests that the elasticity of substitution thereby determines the importance of the benefit side of the prices versus quantities discussion. We now initiate an investigation of this secondary interpretation by reviewing the two extreme cases with which we motivated this entire chapter.

We argued in Section 5.1 that when K and q are perfect substitutes, the output of x will remain constant even as q varies. This conclusion is true, of course, regardless of the source of the variation in q, and thus, regardless of the mode of control placed on q. The comparative advantage of prices would therefore be totally void of a benefit side:

$$\Delta(\rho=1) = \frac{1}{2} C_{11} (\mathrm{Var}(\tilde{q}(\theta,\xi))+\mathrm{Var}(\hat{q}_d(\xi))) - \mathrm{Cov}(\alpha(\theta,\xi);\hat{q}_d(\xi)).$$

Only the last term can be negative and that only when the marginal cost schedule and the output distortion are positively correlated. The first term registers the always positive net bias of the efficiency gain under prices combined with the increase in expected costs due to the resulting variation in output. There is no counterbalancing efficiency gain under quantities, so the second variance term, the increase in expected costs due to output variation under quantities, is also a positive bias toward prices. $\Delta(\rho=1)$ is therefore quite likely to be positive, especially when C_{11} is large. For our present purposes, however, this observation is overshadowed by the result that perfect substitution of inputs has caused the benefit side of the comparative advantage to disappear.

The fixed coefficients case was also noted in Section 5.1; recall that when the elasticity of substitution is zero, $x = (1-\gamma)q$. The comparative advantage of prices under these circumstances is

$$\Delta(\rho = \infty) = \frac{1}{2} B_{11} (1-\gamma)^2 [\text{Var } \tilde{q} - \text{Var } \hat{q}_d] + (1-\gamma) \text{Cov}(\beta(\eta);\hat{q}_d - \tilde{q}) + \frac{1}{2} C_{11} (\text{Var } \tilde{q} + \text{Var } \hat{q}_d) - \text{Cov}(\beta(\eta);\hat{q}_d).$$

The benefit side has been modified by powers of $(1-\gamma)$, and since $(1-\gamma) < 1$, the importance of the benefit side is still diminished. This decrease, to be sure, is the result of the translation of output variation in q to smaller output variation in x; i.e., a one unit change in the production of q will cause a change in the production of x of less than one unit. Both interpretations clearly show promise, but the troublesome intermediate cases are still to be considered.

The difficulty with the intermediate cases lies not in the determination of the comparative advantage of prices, but rather in the determination of the effect on the comparative advantage of a change in the elasticity of substitution. The previous analysis is valid for an arbitrary value of ρ, and thus for an arbitrary value of σ, but these values were specified at the outset. The approximations that we performed require that these values, once specified, remain fixed. We now call that the value ρ_o, to emphasize that the points around which the approximations were made clearly depend on ρ_o. As we now change the elasticity of substitution, we continue to use the previous approximations and require, for the sake of comparison, that the change be affected so that the optimal level of q-production is held at $\hat{q}_o$. The K-response for the x-firm to $\hat{q}_o$, for any ρ, is then

$$\hat{K}_{o\rho} = (\bar{C}'/\bar{r})^{\sigma(\rho)}\ \hat{q}_o \neq \hat{K}_o \qquad (\sigma(\rho) = (1/1-\rho),$$

and the output of x becomes

$$\hat{x}_o = \hat{q}_o(\gamma(\bar{C}'/\bar{r})^{\rho\sigma(\rho)} + (1-\gamma)^{1/\rho} \equiv \hat{q}_o\ D(\rho).$$

The optimal price order, however, remains $\tilde{p} = C'$, so that

$$\tilde{q}(\theta,\xi) = \hat{q}_o - (\alpha(\theta,\xi)/C_{11}),$$

$$\tilde{K}_\rho(\theta,\xi) = (\bar{C}'/\bar{r})^{\sigma(\rho)}\ \tilde{q}(\theta,\xi), \text{ and}$$

$$\tilde{x}_\rho(\theta,\xi) = \tilde{q}(\theta,\xi)\ D(\rho).$$

The optimal quantity order under the output distortion is similarly $\hat{q}_p = \hat{q}_o - E(\phi(\xi))$, so that

$$\hat{q}_d = \hat{q}_p + \phi(\xi);$$

$$\hat{K}_\rho(\xi) = (\bar{C}'/\bar{r})^{\sigma(\rho)}\ \hat{q}_d(\xi); \text{ and}$$

$$\hat{x}_\rho(\xi) = \hat{q}_d(\xi)\ D(\rho).$$

The comparative advantage can now be expressed in exactly the same form as before, but it stands valid for any elasticity of substitution in the neighborhood of $(1/(1-\rho_o))$:

$$\Delta(\rho/\rho_o) = \frac{1}{2} B_{11} (D(\rho))^2 (Var(\tilde{q}(\theta,\xi)) - Var(\hat{q}_d(\xi))$$

$$+ D(\rho) Cov(\beta(\eta); \hat{q}_d - \tilde{q}) - Cov(\alpha(\theta,\xi); \hat{q}_d)$$

$$+ \frac{1}{2} C_{11} (Var(\tilde{q}(\theta,\xi)) + Var(\hat{q}_d(\xi))). \qquad (5.3.3)$$

It remains, therefore, only to compute the effect on $D(\rho)$ of a change in ρ:

$$\frac{\partial D(\rho)}{\partial \rho} = - (1/\rho^2(1-\rho))D(\rho) [\ell n(\gamma(\bar{C}'/\bar{r})^{\rho\sigma(\rho)} + (1-\gamma))]$$

$$\cdot(\gamma(\bar{C}'/\bar{r})^{\rho\sigma(\rho)}) [\ell n(\bar{C}'/\bar{r})]. \qquad (5.3.4)$$

The sign of equation (5.3.4), and thus the direction of the effect of the change in ρ, depends crucially on the signs of the logarithmic terms. Table 5.1 summarizes these signs for the various cases.

Table 5.1

The Sign of $(\partial D(\rho)/\partial\rho)$

	$(\bar{C}'/\bar{r})>0$		$(\bar{C}'/\bar{r})<0$	
Term	$0<\rho<1$	$\rho < 0$	$0<\rho<1$	$\rho < 0$
$\ell n(\gamma(\bar{C}'/\bar{r})^{\rho\sigma(\rho)}+(1-\gamma))$	(+)	(-)	(-)	(+)
$\ell n(\bar{C}'/\bar{r})$	(+)	(+)	(-)	(-)
$(\partial D(\rho)/\partial\rho)$	(-)	(+)	(-)	(+)

We can infer from the table that when $0 < \rho < 1$, an increase in the elasticity of substitution will cause a decrease in the $D(\rho)$ coefficient. The output effect of variation in the deliveries of the intermediate good are seen to diminish. This conclusion is quite consistent with our observation that perfect substitutes allow $\hat{q}_o$ to be produced in all states of nature. We can certainly view this as a decrease in the importance of the benefit side, ceteris paribus, and note that to the extent to which the cost side tends to favor prices,[7] it creates a positive bias for price regulation. The opposite conclusion is drawn when $\rho < 0$. The output effect of delivery variation increases from the factor of $(1-\gamma)$ when K and q are non-substitutable, and the importance of the benefit side increases. The K-response of the x-firm accentuates, rather than alleviates, the effects of q-variation in this case when $\rho < 0$.

The output effect of delivery variation obviously reaches a maximum in the Cobb-Douglas case in which[8]

$$x = (\gamma C'/(1-\gamma)r)^{\gamma}\ q. \qquad (5.3.5)$$

The variation in deliveries of the intermediate good is therefore exaggerated in variation of x when $\gamma C' > (1-\gamma)r$, reduced when $\gamma C' < (1-\gamma)r$, and transferred intact when $\gamma C' = (1-\gamma)r$. We note that it is thereby

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[7]We argued several paragraphs above that it is quite likely that the cost side be positive.

[8]Equation (5.3.5) can be derived directly by noting that

$$\lim_{\rho\to 0} \ell n(D(\rho) = \lim_{\rho\to 0} (1/\rho)\ \ell n\ (\gamma(\gamma C'/(1-\gamma)r)^{\rho\sigma(\rho)} + (1-\gamma))$$

$$= \gamma \ell n\ (\gamma C'/(1-\gamma)r)$$

(using L'Hospital's rule). Thus

$$\lim_{\rho\to 0} D(\rho) = (\gamma C'/(1-\gamma)r)^{\gamma}.$$

possible that the variation in the output of the final good will never exceed that of the intermediate good, regardless of the elasticity of substitution of the process that creates x from q.

We should also note, before terminating this section, that all of the conclusions that were developed in the previous chapters concerning the extreme values of B_{11} and C_{11} are still valid. The comparative advantage when the profit maximizing producer of x uses the wrong price distribution is also a familiar result; the benefit side would then contain both the second moment and the relevant covariances measured around the incorrectly computed mean. Neither of these final observations is surprising, so they are recorded without further justification.

Section 5.4: The Role of Inventories

No discussion of a vertical production process would be complete without at least a brief discussion of inventories and their effects. Their influence, in our discussion, is to reduce the variation in the amount of q delivered to the x-firm, thereby reducing the variation in the output of x, itself. It would, in fact, be possible to maintain the store of q so that $\hat{q}_o$ could be delivered in all states of nature. That store could be maintained indefinitely, since

$$E(\hat{q}_o - \frac{\alpha(\theta,\xi)}{C_{11}}) = E(\hat{q}_o - E\phi + \phi(\xi)) = \hat{q}_o.$$

In this extreme case, x_o would always be produced regardless of the output of the q-firm, and variation in the production of q would be reflected in the cost side alone. This circumstance was discussed fully in the perfect substitution case of Section 5.3. We should also note, in passing, that since $\hat{q}_o$ would be delivered in all states of nature, the x-firm is assured that expected costs will never exceed expected revenues, even under expected marginal value pricing.

The maintenance of positive inventories imply, of course, a loss in foregone consumption, and it is unlikely that levels sufficient to guarantee a constant delivery of $\hat{q}_o$ would be optimal. To the extent that inventories at any level diminish the variation in deliveries, however, they diminish the importance of the benefit side of the comparative advantage of prices. As a rule, therefore, prices become more preferred, or quantities less preferred, as the level of inventories increases. Observe finally that an increase in the number of imperfectly correlated firms producing q will create the very same effect. The decrease in the variation of total output of q as that number increases was a primary result of the third chapter.

Section 5.5: Conclusions

We have seen that the elasticity of substitution affects only the degree to which the variation of the intermediate good is translated into the variation in the output of the final good. Rather than favoring one mode of control over the other, an increase (e.g.) in that elasticity will increase or decrease the importance of the benefit side of the comparison of ρ is less than or greater than zero. Implicit in this result is the corollary that maximum translation of variation occurs when the elasticity of substitution is unity.

The profitability of producing the final good was a secondary interest because negative expected profits could create pressure for attempting to avoid any regulation of the input by integrating its production into the final production process. Pricing policies were demonstrated, however, that both eliminate this difficulty and leave our analysis otherwise intact.

The role of inventories was also briefly studied. To the degree that inventories lessen the variation in input deliveries, the importance of the benefit side of the price-quantities comparison is diminished. An increase in the number of imperfectly correlated firms producing the intermediate good was found to produce the same result. This final observation leans heavily upon the conclusions of the third chapter.

Chapter Six

CONCLUDING REMARKS

The presumption that price controls are generally better than quantity controls has been shown to be a potentially serious error in judgement. The relative merits of placing either a singular price order or a singular quantity order on a particular set of goods are determined by both the magnitude and the nature of the output variation that is created by these two possible modes of control. The impact of this variation on social welfare, and thus the importance of the prices-quantities comparison, depends upon the curvature and interdependence parameters of the cost and benefit functions for the goods in question. We have clearly demonstrated the existence of circumstances in which price controls are significantly inferior to quantity controls. The reverse case is, of course, also quite possible.

We have asked a very special second best question: in the face of uncertainty, if a regulating agency can issue either a single price order or a single quantity order, which one should it choose to maximize expected social welfare? The singular, once-and-for-all character of this question has allowed us to focus our analytic attention upon those properties of prices and quantities that have a direct bearing on their relative value as output controls. To the extent that agencies seek singular simplicity in their control orders, our analysis is applicable to several arenas of current policy debate in this country; our casual automotive illustration certainly suggests one timely example. It is, however, this same speciality that limits the theoretical range of our present analysis. We close our study, therefore, with a cursory catalogue of the directions in which further research can proceed, and thereby place this study in its proper theoretical perspective.

Observe initially that we have not yet developed a model that adequately handles the general pollution problem which motivated our discussion in the Introduction. To do so, however, we need only to envision such a pollutant to be a positive input into the production of a final good, as well as a negative entry into the benefit function. Other inputs would then be thought to substitute for the pollutant in the production process, and the elasticity of substitution would register the ease or difficulty with which the pollutant could be controlled. In terms of our original example, then, we would consider affluent scrubbers to be capital substitutes for the sulfur dioxide emissions of a coal-burning power plant. If we insert the intermediate good of the Chapter Five model into the benefit function, we shall have constructed an analytic framework for which this general pollution case is accessible.

A second, more fundamental limitation of our analysis is its partial equilibrium nature. In specifying a cost function, for instance, we tacitly assume that the producers of the goods to be controlled exert no influence on the factor prices of their inputs. This, of course, need not be the case, and the validity of our results in general equilibrium should be tested before they are applied to any large sectors of an economy. An important arena in which this test could be conducted as part of a dual research project would be a general equilibrium comparison of tariffs and quotas under international uncertainty.

Ours has also been a very static analysis. The type of control to be imposed, as well as its chosen level, was determined by the center before the production period began. The center was assumed to be unable to change either the value or the genre of that control as the period progressed, even though the actual state of the world could have been determined (or at least the number of possibilities limited). While this may not be a realistic framework, it did accentuate the differences between prices and quantities by letting the consequences of each run their entire course. Were we to allow the center an adjustment process for each mode, we should expect the differences in outcomes between the two modes to lessen; the social impact of the prices-quantities choice, therefore, would also be correspondingly diminished.

A more significant restriction is the center's choice of either a single price control or a single quantity control. While this set of choices allowed us to investigate the general preference for prices, it certainly need not contain the best feasible regulatory scheme. Several recent papers, notably a pollution control study by Michael Spence and Marc Roberts,[1] suggest mixed schemes that are preferred to either price controls or quantity controls. Inasmuch as such schemes are discrete steps toward the first best continuous contingency orders, these are not surprising results. They are, nonetheless, significant steps toward a generalization that should be made. A careful search of some simple classes of functions on prices-quantities space may pay high dividends

[1] Marc Roberts and Michael Spence, "Effluent Charges and Licenses under Uncertainty."

in this regard. We should not rule out, however, the potential existence of circumstances in which one of our present two choices is the optimal control.

Finally, recall that the periphery was assumed to be entirely risk neutral, simply seeking to maximize profits. Were this not the case, little would change in the prices-quantities comparison. Output variation under prices would still emerge as one of the crucial determinants; it simply would not have the convenient functional form that is displayed when the peripheral firms are all profit maximizers. As soon as we give the peripheral decision maker a utility function and allow him to react in his own best interest, however, we open the possibility of constructing an incentive function that is socially superior to either mode of control discussed here, and perhaps even to the previously suggested mixes. A search for an optimal incentive function in the context of uncertainty could produce a useful and potentially significant generalization.

Appendix A

PROOF OF LEMMA 4

We will demonstrate the case in which $\alpha(\theta,\xi) > 0$, first. Selecting $(\bar{\theta},\bar{\xi})$ arbitrarily and drawing AB parallel to CD in Figure (A.1), we see that $\overline{DE} = \overline{AE}$ and $\overline{DE} = (\overline{CE}/\tan \pi)$. We know that $\overline{CE} = \alpha(\bar{\theta},\bar{\xi})$, $\overline{AE} = -g(\alpha(\theta,\xi),\tilde{p})$ and $\tan(\pi(\bar{\theta},\bar{\xi}))$ is the slope of marginal costs at $(\hat{q}_o + g(\alpha(\theta,\xi),\tilde{p}))$. Note that marginal costs have their smallest slope at $q = 0$, independent of (θ,ξ) and define M' to be the value of the slope at that point $(C_{11} - \frac{1}{2} C_{111}\, \hat{q}_o)$. Then, since $(\bar{\theta},\bar{\xi})$ is arbitrary, for any (θ,ξ) such that $\alpha(\theta,\xi) > 0$,

$$\begin{aligned}(-g(\alpha(\theta,\xi),\tilde{p})) &= \overline{AE} \\ &\leq \overline{DE} \\ &= (\alpha(\theta,\xi)/\tan \pi) \\ &\leq (\alpha(\theta,\xi)/M').\end{aligned}$$

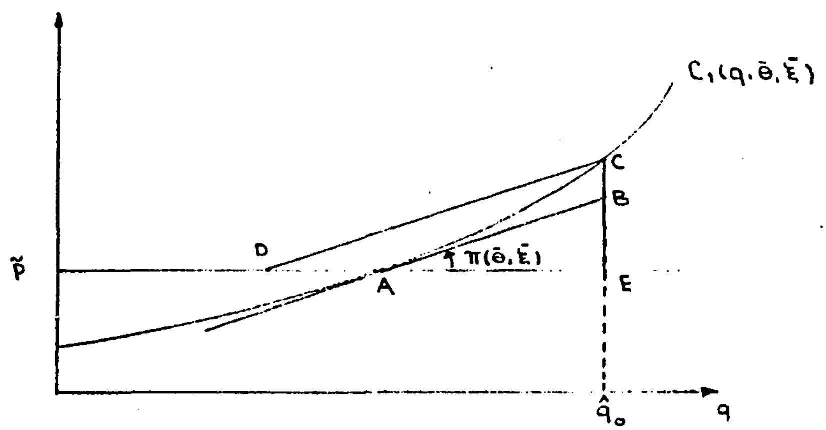

Figure (A.1)

On the other hand, suppose that $(\bar{\theta},\bar{\xi})$ is arbitrarily chosen such that $\alpha(\bar{\theta},\bar{\xi}) < 0$; the output disturbance is then positive. Notice in Figure (A.2) that $\overline{EB} = \overline{ED}$ and $\tan(\delta(\bar{\theta},\bar{\xi}))$ is the slope of marginal cost at the point $(\hat{q}_o + g(\alpha(\bar{\theta},\bar{\xi}),\tilde{p}))$. Therefore,

$$g(\alpha(\bar{\theta},\bar{\xi}),\tilde{p}) = \overline{EB} \geq \overline{ED} = -(\alpha(\bar{\theta},\bar{\xi})/\tan(\delta(\bar{\theta},\bar{\xi}))).$$

The highest value $\tilde{p}$ can assume is the least upper bound of $B_1(0,\eta)$ over all states of nature indexed by η. We certainly do not lose economic generality in assuming that that bound is finite. Define q^{max} to be the point where the marginal costs in the least costly state of nature equal this upper bound and observe that the slope of marginal costs arrives at its relevant maximum at that point. If we define M'' to be that slope $(C_{11} + \frac{1}{2} C_{111}(q^{max} - \hat{q}_o))$, we conclude that for any (θ,ξ) such that $\alpha(\theta,\xi) < 0$,

$$g(\alpha(\theta,\xi),\tilde{p}) \geq (-\alpha(\theta,\xi)/\tan\delta(\theta,\xi)) \geq -\alpha(\theta,\xi)/M''.$$

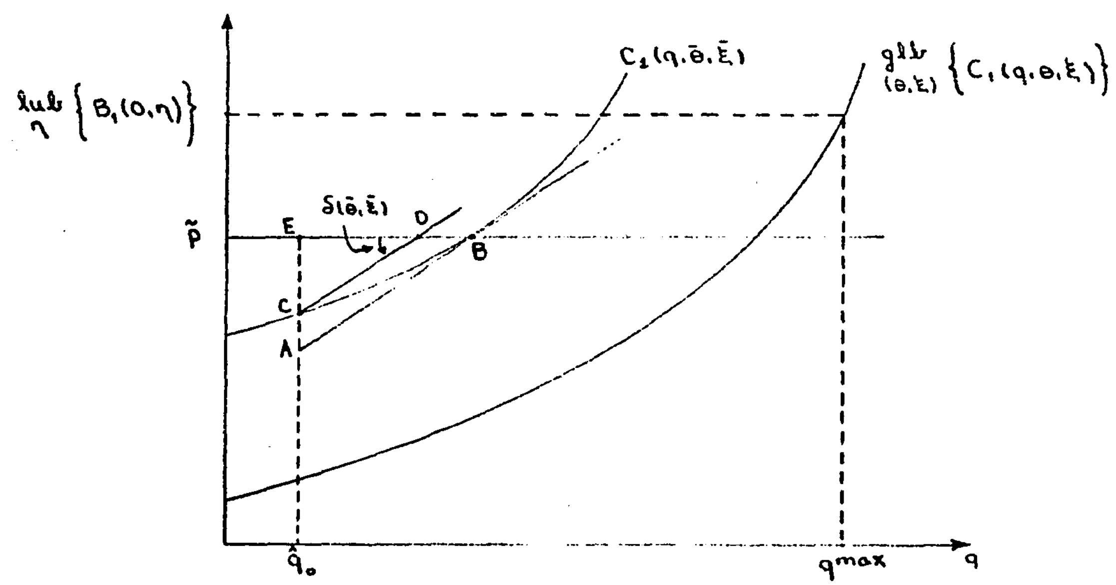

Figure (A.2)

Appendix B

In order to determine the ($\bar{q}_{oi}$: i = 1, ..., n) as defined in Subsection 3.1.2, we must solve the following system of equations:

$$\alpha_1(\theta_1) + C^1_{11}(\bar{q}_{o1}-\hat{q}_{o1}) = \alpha_i(\theta_i) + C^i_{11}(\bar{q}_{oi}-\hat{q}_{oi}) \quad \forall i=2,\ldots,n.$$

$$\sum_{i=1}^{n} (\bar{q}_{oi} - \hat{q}_{oi}) = 0$$

Writing this system in matrix notation,

$$\begin{vmatrix} C^1_{11} & -C^2_{11} & 0 & \cdots & 0 \\ C^1_{11} & 0 & -C^3_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ C^1_{11} & 0 & \cdots & 0 & -C^n_{11} \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix} \begin{vmatrix} \bar{q}_{o1} - \hat{q}_{o1} \\ \bar{q}_{o2} - \hat{q}_{o2} \\ \vdots \\ \\ \bar{q}_{on} - \hat{q}_{o2} \end{vmatrix} = \begin{vmatrix} \alpha_2(\theta_2) - \alpha_1(\theta_1) \\ \alpha_3(\theta_3) - \alpha_1(\theta_1) \\ \vdots \\ \alpha_n(\theta_n) - \alpha_1(\theta_1) \\ 0 \end{vmatrix}$$

so that by Cramer's Rule, we see that

$$\bar{q}_{oi} - \hat{q}_{oi} = \begin{vmatrix} C^1_{11} & -C^2_{11} & \cdots & (\alpha_2-\alpha_1) & \cdots & 0 \\ C^1_{11} & 0 & \cdots & (\alpha_3-\alpha_1) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ C^1_{11} & 0 & \cdots & (\alpha_i-\alpha_1) & \cdots & 0 \\ & & & \vdots & & \\ C^1_{11} & 0 & \cdots & (\alpha_n-\alpha_1) & \cdots & C^n_{11} \\ 1 & 1 & \cdots & 0 & \cdots & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} C^1_{11} & -C^2_{11} & \cdots & 0 \\ C^1_{11} & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ C^1_{11} & 0 & & -C^n_{11} \\ 1 & 1 & \cdots & 1 \end{vmatrix} \tag{A.1}$$

Induction on the number of firms is required to compute the determinants in equation (A.1); we will tackle the denominator first. To simplify the calculation, we assume that $C^i_{11} = C^j_{11}$ for all i and j. We quickly verify that for n = 2 and n = 3, the denominator equals $nC_{11}^{(n-1)}$. We therefore assume that for n = m,

$$\begin{vmatrix} C_{11} & -C_{11} & \cdots & 0 \\ C_{11} & 0 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ C_{11} & 0 & \cdots & -C_{11} \\ 1 & \cdots & \cdots & 1 \end{vmatrix} = mC_{11}^{m-1},$$

and seek an expression for the solution when n = (m+1) on the basis of that assumption. By rearranging columns and rows, we see that

$$\begin{vmatrix} C_{11} & -C_{11} & \cdots & 0 \\ C_{11} & 0 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{vmatrix} = (-1)^{m-1} \begin{vmatrix} C_{11} & \cdots & \cdots & -C_{11} \\ C_{11} & -C_{11} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ C_{11} & \cdots & -C_{11} & 0 \\ 1 & & 1 & 1 \end{vmatrix}$$

$$= (-1)^{m-1} C_{11} \begin{vmatrix} -C_{11} & 0 & \cdots & 0 \\ \vdots & -C_{11} & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{vmatrix} + (-1)^{m+1}(-1)^{n-1} C_{11} \begin{vmatrix} C_{11} & -C_{11} & \cdots & 0 \\ C_{11} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{vmatrix}$$

$$= (-1)^{m-1} (-1)^{m-1} C_{11}^m + (-1)^{2m} C_{11} (m\, C_{11}^{m-1})$$

$$= (m + 1)\, C_{11}^m.$$

We can now attack the numerator in like fashion. For n = 2,

$$\begin{vmatrix} C_{11} & (\alpha_2-\alpha_1) \\ 1 & 0 \end{vmatrix} = (\alpha_2 - \alpha_1) = \begin{vmatrix} (\alpha_2-\alpha_1) & C_{11} \\ 0 & 1 \end{vmatrix}$$

For n = 3, there are three possible cases:

(1): $$\begin{vmatrix} C_{11} & (\alpha_2-\alpha_1) & 0 \\ C_{11} & (\alpha_3-\alpha_1) & -C_{11} \\ 1 & 0 & 1 \end{vmatrix} = C_{11}\ (\alpha_3+\alpha_1-2\alpha_2)$$

(2): $$\begin{vmatrix} (\alpha_2-\alpha_1) & -C_{11} & 0 \\ (\alpha_3-\alpha_1) & 0 & -C_{11} \\ 0 & 1 & 1 \end{vmatrix} = C_{11}\ (\alpha_2+\alpha_3-2\alpha_1)$$

(3): $$\begin{vmatrix} C_{11} & -C_{11} & (\alpha_2-\alpha_1) \\ C_{11} & 0 & (\alpha_3-\alpha_1) \\ 1 & 1 & 0 \end{vmatrix} = C_{11}\ (\alpha_1+\alpha_3-2\alpha_3)$$

Generalizing these results, we assume that for n = m and for any i that the determinant equals

$$\left[\sum_{\substack{k=1\\k\neq i}}^{m} \alpha_k(\theta_k) - (m-1)\ \alpha_i(\theta_i)\right] C_{11}^{m-2}$$

and question its value when n = (m+1):

$$\begin{vmatrix} C_{11} & -C_{11} & \overset{(i)}{(\alpha_2-\alpha_1)} & \cdots & 0 \\ C_{11} & 0 & \vdots & & \vdots \\ \vdots & \ddots & (\alpha_i-\alpha_1) & & \vdots \\ \vdots & & \vdots & \ddots & -C_{11} \\ 1 & 1 & \cdots\cdots\ 0 & \cdots\cdots & 1 \end{vmatrix} = (-1)^{m-1} \begin{vmatrix} C_{11} & \cdots\cdots & \overset{(i)}{\alpha_{m+1}-\alpha_1} & \cdots\cdots & -C_{11} \\ C_{11} & -C_{11} \cdots & \alpha_2-\alpha_1 & \cdots\cdots & 0 \\ C_{11} & \cdots\cdots & \alpha_i-\alpha_1 & \cdots\cdots & 0 \\ C_{11} & \cdots\cdots & \alpha_m-\alpha_1 & \cdots -C_{11} & 0 \\ 1 & \cdots\cdots & 0 & \cdots\cdots\ 1 & 1 \end{vmatrix}$$

$$= (-1)^{m-1}\{C_{11} \begin{vmatrix} -C_{11} \cdots & \overset{(i)}{\alpha_2-\alpha_1} & \cdots\cdots & 0 \\ 0 & \vdots & & \vdots \\ \vdots\ \ddots & \alpha_i-\alpha_1 & & \vdots \\ \vdots & \vdots & \ddots\ -C_{11} & 0 \\ 1 \cdots & 0 & \cdots\cdots & 1 \end{vmatrix} + (-1)^{i-1}(\alpha_{m+1}-\alpha_1) \begin{vmatrix} C_{11} & -C_{11} & \cdots\cdots & & 0 \\ \vdots & \ddots & -C_{11} & & \vdots \\ C_{11} & \cdots\cdots & 0 & 0 & \vdots \\ \vdots & & \vdots & -C_{11} & \vdots \\ \vdots & & \vdots & \ddots\ C_{11} & \vdots \\ 1 & \cdots\cdots & \cdots & \cdots\cdots & 1 \end{vmatrix}$$

$$+ (-1)^{m+1}C_{11}[\sum_{\substack{k=1 \\ k\neq i}}^{m} \alpha_k(\theta_k)-(m-1)\alpha_i(\theta_i)]C_{11}^{m-2}\} \qquad (A.2)$$

The first determinant in (A.2) equals $(-1)^{m-2}C_{11}^{m-2}(\alpha_i-\alpha_1)$, since the $(\alpha_2-\alpha_1)$-submatrix has a row of zeros. The second matrix is a bit stickier. An induction argument similar to that used to compute the denominator reveals, however, that it equals $(-1)^{m-i}C_{11}^{m-1}$. The expression on the right hand side of (A.2) therefore reduces to

$$\{(-1)^{2m-3}C_{11}^{m-1}\ (\alpha_i-\alpha_1) + (-1)^{4m-4}(-1)^{-2}\ C_{11}^{m-1}\ (\alpha_{m+1}-\alpha_1)$$

$$+ (-1)^{2m}\ C_{11}^{m-1}\ [\sum_{\substack{k=1 \\ k\neq i}}^{m} \alpha_k(\theta_k) - (m-1)\ \alpha_i(\theta_i)]\ C_{11}^{m-2}\}$$

$$= [\sum_{\substack{k=1 \\ k\neq i}}^{m+1} \alpha_k(\theta_k) - (m)\ \alpha_i(\theta_i)]\ C_{11}^{m-1}.$$

Thus, we have shown that the numerator of (A.1) is precisely

$$\Big[\sum_{\substack{k=1\\k\neq i}}^{m+1} \alpha_k(\theta_k) - (n-1)\,\alpha_i(\theta_i)\Big]\, C_{11}^{n-2}$$

for any i and, as a result, we have shown that

$$(\bar{q}_{oi}-\hat{q}_{oi}) = \Big[\sum_{\substack{k=1\\k\neq i}}^{n} \alpha_k(\theta_k) - (n-1)\,\alpha_i(\theta_i)\Big]/nC_{11}.$$

for any $i = 1, \ldots, n$.

Appendix C

We are required to prove the following corollary when the correlation coefficient across identical firms is positive.

Corollary 1 (page 114):

Suppose that quantity controls are preferred when the industry is taken as a whole. There exists a subset of $\bar{m}$ firms such that a mix that controls those $\bar{m}$ firms by prices and the remaining $(n-\bar{m})$ firms by quantities is favored to industry-wide quantity control if and only if

$$(1/n)|B_{11}/C_{11}| \leq 1 \tag{3.2.5}$$

Recall that we have defined

$$\Delta(m/n) \equiv (m/n)\{\rho[\frac{m}{n}(\frac{B_{11}\sigma^2}{2}) + \frac{\sigma^2}{2}] + (1-\rho)\ [\frac{1}{n}(\frac{B_{11}\sigma^2}{2}) + \frac{\sigma^2}{2}]\}$$

to be the comparative advantage of prices over quantities for a subset of m firms taken in the context of the cumulative position in the industry (ρ is the correlation coefficient of output variation under prices across firms). Then

$$\begin{aligned} &(1/n)|B_{11}/C_{11}| \leq 1 \\ &\quad \Longleftrightarrow (1 + (1/n)(B_{11}/C_{11})) \geq 0 \\ &\quad \Longleftrightarrow (1/n)(\frac{\sigma^2}{2} + \frac{1}{n}\ (\frac{B_{11}\sigma^2}{2})) \equiv \Delta(1/n) \geq 0 \end{aligned}$$

We conclude, therefore, that

$$(1/n)|B_{11}/C_{11}| > 1$$

implies that $\Delta(1/n) \leq 0$. Observing that

$$\Delta(m/n) \leq \Delta(1/n) \; ; \quad \forall\, m \geq 1$$

when $\rho > 0$, we see that

$$(1/n)|B_{11}/C_{11}| > 1$$

implies immediately that

$$\Delta(m/n) \leq \Delta(1/n) \leq 0,$$

as well. There cannot exist any subsets in which prices are preferred as long as $(1/n)|B_{11}/C_{11}| > 1$.

Conversely, when $(1/n)|B_{11}/C_{11}| \leq 1$, then $\Delta(1/n) \geq 0$, so that there exists at least the singleton subset for which prices are preferred. Sufficiency is therefore trivially confirmed. The optimal number of firms in the price controlled subset, $\bar{m}$, can easily be shown to be the integer closest to

$$\left(\frac{\rho-1-n(C_{11}/B_{11})}{\rho}\right).$$

Bibliography

Dewes, Donald. Economics and Public Policy: The Automobile Pollution Case. Cambridge, Mass.: MIT Press, 1974.

Hayek, F. A. "The Use of Knowledge in Society." American Economic Review, Vol. 35 (September 1935), pp. 519-530.

Heal, G. M. "Planning Without Prices." Review of Economic Studies, Vol. 36 (July 1969), pp. 347-362.

Henderson, James and Richard Quandt. Microeconomic Theory. New York: McGraw-Hill, Inc., 1971.

Hoel, Paul G., Sidney C. Port, and Charles Stone. Introduction to Probability Theory. Boston: Houghton Mifflin Co., 1971.

Jacoby, Henry, John Steinbruner, et. al. Clearing the Air: Federal Policy on Automotive Emissions Control. Cambridge, Mass.: Ballinger Publishing Company, 1973.

Marglin, S. A. "Information in Price and Command Systems of Planning," in Margolis, J. (ed.), Activity Analysis in the Theory of Growth and Planning. New York: MacMillan, Inc., 1967, pp. 54-77.

Motor Vehicle Manufacturers Association of the U.S., Inc. 1973 Automobile Facts and Figures. Detroit: 1974.

Poole, William. "Optimal Choice of Monetary Policy Instruments in a Simple Stochastic Macro Model." Special Studies Paper Number 2, Federal Reserve Board, 1969.

Ross, Stephen. "Portfolio and Capital Market Theory with Arbitrary Preferences and Distributions--The General Validity of the Mean-Variance Approach in Large Markets." University of Pennsylvania Working Paper No. 12-72, 1974.

Samuelson, Paul A. "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variance, and Higher Moments." Review of Economic Studies, Vol. 37 (October 1970), pp. 537-542.

Spence, Michael and Marc Roberts. "Effluent Charges and Licenses Under Uncertainty." Stanford University Technical Report No. 146 (October 1974).

Springer, George and Donald Patterson. Engine Emissions. New York: Plenum Press, 1973.

Stern, Arthur (ed.). Air Pollution: Sources of Air Pollution and Their Control. New York: Academic Press, 1968.

Tobin, James. "The Theory of Portfolio Selection," in Hahn, F. H. and F. P. R. Brechling (eds.), The Theory of Interest Rates. London: MacMillan Press, 1965.

Weitzman, Martin. "Prices and Quantities." The Review of Economic Studies, Vol. 41 (October 1974), pp. 477-493.

________. "Iterative Multilevel Planning with Production Targets." Econometrica, Vol. 38 (January 1970), pp. 50-65.

Whinston, A. "Price Guides in Decentralized Institutions." Unpublished Ph.D. Dissertation, Carnegie, 1962.