The complementarity of public and private capital and the optimal rate of return to government investment*

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The impacts of various imperfections in the capital market on the optimal return to government investment have been the subjects of intense investigation in the recent literature. The distortion caused by corporate profits taxation has, for example, been extensively studied by Baumol, Sandmo and Dreze, Seagraves, and others.1 They have argued that under 50 percent taxation, the proper discount rate for public investment is some weighted average of the return to a riskless government bond (say r) and the return to private capital (2r in this case). Others have argued, in another context, that to the extent that a risk premium on private investment is a private cost and reflects the imperfect spreading of risk by a market, the government should ignore it; the pure rate of time preference is the appropriate discount factor.2 These are, of course, contradictory answers to what is formally the same question: how should the government react when the private sector makes its investment decisions in a distorted capital market?

We shall find the key to the apparent confusion in different characterizations of the underlying production structure. Those who

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have studied the tax impact have assumed, at least implicitly, that public and private capital are positive substitutes in the Hicksian q-sense: increases in the private (public) capital stock reduce the marginal product of public (private) capital. Those who argue for ignoring the private risk premium have meanwhile presumed that the two types of capital are, on the margin, independent. While these may be reasonable approximations of reality in some cases, they exclude one important dimension of government investment. There are, in particular, many examples in which public and private capital actually complement each other. The construction of an improved highway system, for instance, often significantly increases the productivity of private industry in a specified region. An irrigation project will usually increase the agricultural yields of the areas it services. A subsidy to scientific research or education by the government can be considered investment that can raise the productivity of private capital. The list is virtually endless. The very reason that the government chooses to intervene in these cases lies in the external economies that such activities can create; the technical assumptions of the previous studies that preclude them are simply too restrictive.

In the present paper, therefore, we provide a general characterization of the socially optimal rate of return of government investment. In so doing, we shall acknowledge all of the possible interactions between public and private capital in any economy with a distorted capital market. The private sector will be presumed to invest up to the point at which the marginal return to such investment equals an arbitrary constant. This constant can reflect the corporate profits tax, a private risk premium, or any of the other distortions that might be influencing the private investment decision.

Section I specifies the model in which we shall be working, and the prescribed characterization is derived. It is found that if the requisite rate of return to private investment exceeds the pure rate of time preference, for example, then the optimal marginal return to government investment can be greater than, equal to, or less than the rate of time preference as public and private capital are substitutes, independent entities, or complements. The government's best strategy is thus technologically determined. When public and private capital are perfect substitutes, government is helpless to counterbalance a market distortion, and the efficiency of having all capital support equal returns dominates. Its effectiveness in correcting the market distortion, however, increases as the nature of the technological relationship between the two capital types moves from being imperfect substitutes into the realm of complementarity. In that realm, underinvestment by the private sector can be counterbalanced by public
investment that increases the marginal product of private capital; returns to government capital are thus not only less than the return to private capital, but also less than the market rate of interest.

Two subsequent sections will apply the analysis. Section II explicitly incorporates the profits tax in the context of a model of a market economy. We shall utilize an aggregation process that allows the market to be reduced to the form outlined and analyzed in Section I, and apply the general results directly. Our conclusion will be quite familiar when the two types of capital are substitutes, but will be markedly different when they are independent or complementary. Section III then simply notes the impact of differing rates of depreciation on our results.

I. OPTIMAL GOVERNMENT INVESTMENT UNDER DISTORTION

We shall be considering a model in which the life of any publicly provided project is $T$ periods. The economy’s intertemporal welfare function over the lifetime of such projects is then given by

\begin{equation}
U(C_1, \ldots, C_T),
\end{equation}

where $C_i$ is consumption in period $i$. The existing stocks of both public ($G_t$) and private ($P_t$) capital in period $t$ determine total output in the period in accordance with

\begin{equation}
Y_t = \phi(P_t, G_t).
\end{equation}

We assume that $\phi_i > 0$ and $\phi_{ii} < 0$, for $i = P, G$, but the sign of $\phi_{PG}$ reflects whether public and private capital are complements ($\phi_{PG} > 0$) or substitutes ($\phi_{PG} < 0$). We further assume that either type of capital can provide a constant flow of service throughout its lifetime without net investment; thus,

\begin{equation}
G_t = G_{t-1} + \Delta G_{t-1},
\end{equation}

and

\begin{equation}
P_t = P_{t-1} + \Delta P_{t-1},
\end{equation}

where $\Delta G_t$ and $\Delta P_t$ are investment during period $t$ into public and private capital, respectively. These investments are derived from that period’s total output, so that

\begin{equation}
C_t = Y_t - \Delta P_t - \Delta G_t.
\end{equation}

A final assumption produces the private investment decision: private individuals invest to the point where the marginal physical
product of private capital equals a prescribed cutoff point $k_t$. Thus,

$$\phi_P(P_t, G_t) = k_t; \quad t = 1, \ldots, T.$$  

We are not yet concerned with the specific mechanism that determines this behavior. Instead, we simply observe the cutoff and presume that it emerges from decentralized decisions in the capital market. The $k_t$ parameters therefore reflect not only the extent of the distortions that exist in the market, but also the pure rate of time preference. The reader should note, however, that we have made the partial-equilibrium assumption, that the $k_t$ are independent of the level of government investment.

After specifying the initial levels of public and private capital ($G_1$ and $P_1$), we are left only with characterizing the optimal return to governmental investment. Maximizing (1) subject to (2) through (5) leads to the answer; the appropriate Lagrangean is

$$W = U(C_1, \ldots, C_T) + \sum_{t=1}^{T} \mu_t (\phi_P(P_t, G_t) - C_t - \Delta P_t - \Delta G_t)$$

$$+ \sum_{t=1}^{T} \nu_t (P_t + \Delta P_t - P_{t+1}) + \sum_{t=1}^{T} \omega_t (G_t + \Delta G_t - G_{t+1})$$

$$+ \sum_{t=1}^{T} \eta_t (\phi_P(P_t, G_t) - k_t).$$

The appropriate first-order conditions require that

$$W_{C_t} = U_t - \mu_t = 0,$$

$$W_{\Delta P_t} = -\mu_t + \nu_t = 0,$$

$$W_{\Delta G_t} = -\mu_t + \nu_t = 0,$$

$$W_{P_t} = \mu_t \phi_P(P_t, G_t) + \nu_t - \nu_{t-1} + \eta_t \phi_{PP}(P_t, G_t) = 0,$$

and

$$W_{G_t} = \mu_t \phi_{G_t}(P_t, G_t) + \omega_t - \omega_{t-1} + \eta_t \phi_{PG}(P_t, G_t) = 0.$$  

As a result, $\mu_t = \nu_t = \omega_t = U_t$, where $U_t$ denotes $\partial U/\partial C_t$, and (6) and (7) reduce to

$$\eta_t \phi_{PP}(P_t, G_t) = U_{t-1} - (1 + \phi_P(P_t, G_t))U_t,$$

$$\eta_t \phi_{PG}(P_t, G_t) = U_{t-1} - (1 + \phi_G^*(P_t, G_t))U_t.$$  

3. The conditions listed are valid for all periods but the last. We are not concerned with behavior in period $T$, and the requirements for optimality in that period are not recorded.
Combining these by eliminating the shadow price and recalling (5), we find that\(^4\)

\[
\phi^*_G(t) = \left(\frac{\phi_{PG}}{\phi_{PP}}\right)_t k_t + (1 - \left(\frac{\phi_{PG}}{\phi_{PP}}\right)_t) r_t,
\]

where

\[
r_t = \frac{(U_{t-1}/U_t) - 1}.
\]

The optimal marginal return to government investment (\(\phi^*_G(t)\)) is therefore a weighted average of the marginal physical product of private capital (\(k_t\)) and the rate of time preference (\(r_t\)).

This condition is not new, as it stands, but its additional significance becomes clear when we observe that the sign and magnitude of the ratio \(\frac{\phi_{PG}}{\phi_{PP}}\) reflect the technological relationship between public and private capital. To see this point, we rewrite (10) as

\[
\phi^*_G(t) = r_t + \left(\frac{\phi_{PG}}{\phi_{PP}}\right)_t (k_t - r_t).
\]

Since \(\phi_{PP}(t) < 0\), the optimal marginal return to public capital can be greater or less than \(r_t\), depending upon the signs of \(\phi_{PG}(t)\) and \((k_t - r_t)\).

Consider, for example, the case in which imperfect risk spreading in the capital market leads to \(k_t > r_t\) for all \(t\). A misallocation of resources results, and the economy provides too much current consumption at the expense of too little investment for the future. We are now asking how much power the government can muster optimally to counterbalance this underinvestment with its own investment decisions. In the extreme case where public and private capital are perfect substitutes (\(\phi_{PP} = \phi_{PG}\)), a unit increase in public investment simply replaces one unit of private investment; the government is powerless to correct the distortion. Equation (12) tells us that public investment should also return \(k_t\) on the margin. If the two types of capital are independent (as in the additive production function assumed by Arrow and Lind\(^5\)), then \(\phi_{PG} = 0\), and a unit increase in public capital has no effect on the marginal product of private investment. The government should then act as if the distortion in the capital market does not exist and continue to invest until a marginal

\[^4\] Notationally, we write

\[
\phi^*_G(t) \equiv \phi_G(P^*_t, G^*_t),
\]

and

\[
\left(\frac{\phi_{PG}}{\phi_{PP}}\right)_t \equiv \phi_{PG}(P_t, G_t)/\phi_{PP}(P_t, G_t).
\]

Whenever it is unambiguous or immaterial, the time notation is dropped.

\[^5\] Arrow and Lind, op. cit.
return of $r_t$ is achieved. When public and private capital are complements, however, we finally observe that a unit increase in public investment can actually induce an expansion in private investment by increasing its marginal productivity. As a result, public investment becomes an effective vehicle with which to reduce the tendency of the private sector to underinvest. The optimal investment rule therefore calls for public investment to continue beyond the point where its marginal product equals the rate of time preference.

The opposite conclusions are drawn, quite naturally, when the market distortion provides extra incentives to private investment. The intuition developed around the government’s ability to counterbalance the effects of the distortion is, nonetheless, accurate. When there are no distortions, finally, $k_t = r_t$, and the second term of (12) disappears. The allocation problem is then a first-best question, and the optimal marginal return to $G$ is simultaneously equal to the return to private capital and the rate of time preference. We can now summarize our findings in a proposition:

**PROPOSITION IA.** The marginal return to government investment should exceed (fall below) the pure rate of time preference if and only if (i) $P$ and $G$ are substitutes (complements) and the marginal return to private capital exceeds the pure rate of time preference, or (ii) $P$ and $G$ are complements (substitutes) and the pure rate of time preference exceeds the marginal return to private capital.

**PROPOSITION IB.** The marginal return to government investment should equal the pure rate of time preference if and only if (i) $P$ and $G$ are independent, regardless of the cutoff for private investment, or (ii) the marginal return to private capital equals the rate of time preference, regardless of the nature of public and private capital.

**II. AN APPLICATION: CORPORATE INCOME TAXATION**

We now consider the optimal public investment rule for an economy that taxes corporate income and issues government bonds to finance public projects. A constant proportional tax rate $\tau$ is assumed to be applied to all corporate income, and we define the income
of the $i$th firm in period $t$ ($\pi^i_t$) to be its gross output net of its wage bill:

$$\pi^i_t = \psi^i(l^i_t, p^i_t, G_t) - w_t l^i_t, \quad i = 1, \ldots, n.$$  

The labor employment and private capital stock of the $i$th firm in period $t$ are given by $l^i_t$ and $p^i_t$, respectively, while the wage and stock of government capital common to all firms are $w_t$ and $G_t$.

The analysis is simpler if we presume further that each firm is owned by a single individual who makes the corporate investment decision as part of his intertemporal utility maximization. Personal income is therefore derived from three sources. Wage income is generated by supplying $s^i_t$ units of labor services, and totals $w_t s^i_t$. Holdings of risk-free bonds in the amount of $b^i_t$ yield interest income equal to $r_t b^i_t$, where $r_t$ is an exogenously determined return. Holdings of private capital finally return $(1 - \tau)\pi^i_t$. Each individual then allocates this income among current consumption ($c^i_t$), further purchases of government bonds ($\Delta b^i_t$), and further investment in private capital ($\Delta p^i_t$). The budget constraint captures the entire relationship:

$$c^i_t + \Delta b^i_t + \Delta p^i_t = r_t b^i_t + w_t s^i_t + (1 - \tau)\pi^i_t,$$

where, of course, $b^i_t = b^i_{t-1} + \Delta b^i_{t-1}$ and $p^i_t = p^i_{t-1} + \Delta p^i_{t-1}$.

Assuming that $\{r_t\}$ and $\{G_t\}$ are known, we observe that intertemporal utility maximization for each individual requires that

$$u^i_t / u^i_{t-1} = (1 + r_t),$$

$$u^i_{t-1} \psi^i_p(l^i_t, p^i_t, G_t) = (1 - \tau) \frac{\delta \psi^i}{\delta p^i_t} = r_t,$$

and

$$\psi^i_l(l^i_t, p^i_t, G_t) = w_t.$$  

All three rules are familiar. Equation (15) allocates income between current consumption and investment; it requires that investment proceed until the intertemporal marginal rate of substitution equals the return to the riskless bonds. The second rule then allocates investment funds between private and public capital; the return to the last dollar invested in either should be equal after taxes are paid. In the notation of Section I, therefore,

$$k_t = r_t / (1 - \tau).$$

Equation (17) records the conditions for the employment of labor; the
marginal value product of labor should precisely equal the wage rate.

The government must, of course, be fully cognizant of these reactions as it maximizes its intertemporal welfare function (equation (1)) by providing the optimal stream of public investment. The policy objective is to select the stream that will induce the correct stream of private investment in the light of (15), (16), and (17). We must, however, aggregate properly to be able to cast this problem in the analytical framework presented in the previous section.

Aggregate consumption in period $t$ is simply given by the sum of equation (14) over the entire population; that is,

$$C_t = r_tB_t + w_tS_t + (1 - \tau)\Pi_t - \Delta B_t - \Delta P_t,$$

where $C_t = \Sigma_{i=1}^{N} c^i_t$ is aggregate consumption, etc. In order to focus our analysis on the capital market, we presume now that the aggregate supply of labor ($S_t$) is not only inelastic, but also fixed over the planning horizon. If $w_t$ clears the labor market, we have that

$$N \sum_{i=1}^{N} h^i(t) = z(t) - w_t S.$$

As a result,

$$C_t = r_tB_t + (1 - \tau)\sum_{i=1}^{N} \psi^i(t) - w_tS.$$

The return to the riskless bonds in each period will again be the benchmark, and they are assumed to be exogenously fixed in each period. Notice that the demand for new bonds in the private sector ($\Delta B_t$) is determined by (15) and (16) once $\{r_t\}$ and $\{G_t\}$ are given. The government’s supply of new bonds ($\Delta D_t$), however, is determined as the difference between tax revenues and the sum of the debt services ($r_tD_t$) and public investment ($\Delta G_t$); that is,

$$\Delta D_t = r_tD_t + \Delta G_t - \tau \Pi_t.$$

We are therefore overspecifying the bond market, since there is no guarantee that $\Delta B_t = \Delta D_t$ at the given $r_t$. To circumvent this difficulty, it is convenient to presume a perfectly elastic supply of foreign bonds denoted $F_t$ that return $r_t$. These foreign bonds then meet any excess demand for domestic government bonds. The private holdings of bonds are therefore simply the sum of the stocks of domestic and foreign bonds, and excess demand in the bond market is zero for the given $\{r_t\}$. Notationally,

$$B_t = D_t + F_t,$$
and
\[(22) \quad \Delta B_t = \Delta D_t + \Delta F_t.\]

Returning to (19), we can observe that
\[(23) \quad C_t = (1 - \tau) \sum_{i=1}^{N} \psi_i(l_{t,i}p_{t,i}G_t) + r_t F_t - \Delta P_t - \Delta G_t - \Delta F_t;\]
domestic bonds have disappeared from the consumption equation.

The remaining difficulty lies in aggregating the production function. We can assume, however, that the capital market is operating efficiently according to (16) and (17) for any level of government investment. Defining
\[\Psi(S,P_t,G_t) = \left\{ \max \sum_{i=1}^{N} \psi_i(l_{t,i}p_{t,i}G_t) \right\} \left\{ \sum_{i=1}^{N} l_{t,i} \leq S; \sum_{i=1}^{N} p_{t,i} \leq P_t \right\},\]
we can observe that
\[(24) \quad \Psi(S,P_t,G_t) = \sum_{i=1}^{N} \psi_i(l_{t,i}p_{t,i}G_t),\]
where \(l_{t,i}p_{t,i}\) emerge from (16) and (17) for all \(i = 1, \ldots, N\). The function \(\Psi\) is thus maximized when each of its components is maximized. As a result,
\[\Psi_p(t) = \delta\psi / \delta P_t = \psi_p(t) = r_t/(1 - \tau),\]
for all \(i = 1, \ldots, N\) and \(t = i, \ldots, (T - 1)\). The conditions given by (16) can therefore be summarized by
\[(25) \quad \Psi_p(t) = r_t/(1 - \tau).\]

Since aggregate employment is constant over time, we finally see that
\[\Psi(S,P_t,G_t) = \Psi(S,P_t,G_t),\]
where \(\Phi(P_t,G_t) = \Psi(S,P_t,G_t)\).

We can now formalize the government’s optimization problem; it will maximize \(U(C_1, \ldots, C_m)\) with respect to \([G_t]_{t=1}^{T}\) and subject to equation (27),

7. The maximization of \(\Sigma_{i=1}^{N} \psi_{i}(l_{t,i}p_{t,i}G_t)\) subject to \(\Sigma l_{t,i} \leq S\) and \(\Sigma p_{t,i} \leq P_t\) requires the equality of marginal products of labor and private capital across all the firms, both of which are satisfied in our mixed economy due to (16) and (17).

8. By definition (24), \(\delta\Psi / \delta P_t = \Sigma_{i=1}^{N} (\partial\psi / \partial p_{t,i})(\partial p_{t,i} / \partial P_t) = (r_t/(1 - \tau) \Sigma_{i=1}^{N} (\partial p_{t,i} / \partial P_t) = r_t/(1 - \tau).\)
\[ \Phi_P(t) = \frac{r_t}{1 - \tau}, \]
\[ P_t = P_{t-1} + \Delta P_{t-1}, \]
\[ G_t = G_{t-1} + \Delta G_{t-1}, \]
and
\[ F_t = F_{t-1} + \Delta F_{t-1}. \]

Initial values of \( P, G, \) and \( F \) plus \( \{r_t\}_{t=1}^T \) are known. This problem is formally identical to the one analyzed in Section I except for the appearance of foreign bonds. It should be no surprise that the optimal rule for government investment is identical. Substituting
\[ (U_{t-1}/U_t) - 1 = r_t, \]
and
\[ k_t = \frac{r_t}{1 - \tau}, \]
into (10), we find that
\[ (28) \quad \Phi_G^*(t) = r_t + \left( \frac{\Phi_{PG}/\Phi_{PP}}{1 - \tau} \right) r_t. \]

There are several points that should be made in passing. First of all, for a particular size of \( \tau \), the sign of \( \Phi_{PG} \) determines whether the optimal rate of return to public investment is greater or smaller than the return to the riskless bond; the rationale behind this result has been explained for a more general case in Section I. Second, given \( \Phi_{PG} \neq 0 \), the divergence of \( \Phi_G^*(t) \) is larger, the larger the tax rate; i.e.,
\[ \Phi_G^*(t)/r = 1 + \left( \frac{\Phi_{PG}/\Phi_{PP}}{1 - \tau} \right) r_t, \]
and \( \tau/(1 - \tau) \) is an increasing function of \( \tau \).

The reader should also observe that the previously reported results are, in effect, special cases to which (28) can be applied. If, as has usually been assumed, \( \tau = 0.5 \) and \( \Phi_{PG} < 0 \), then \( (\Phi_{PG}/\Phi_{PP})_t > 0 \), and
\[ \Phi_G^*(t) = (1 + (\Phi_{PG}/\Phi_{PP})_t)r_t > r_t. \]
Indeed, when \( P \) and \( G \) are perfect substitutes, the optimal return to public investment is \( 2r_t \). We concur with Baumol and the others, therefore, only when public and private capital are substitutes; in those cases,
\[ r_t < \Phi_G^*(t) < 2r_t. \]
In all other cases, however, \( \Phi^*_G(t) \leq r_t \), with equality holding only when \( P \) and \( G \) are independent.

Finally, equation (28) provides a convenient, though somewhat artificial, framework in which to illustrate how the conclusions are reversed if the governmentally created distortions actually encourage private investment. Suppose that \( \tau \) is, in fact, a subsidy that is aimed at fostering extensive provision of a particular type of private capital. In that case \( \tau/(1-\tau) \) is strictly negative, and the optimal return to government capital is less than \( r_t \) if and only if \( P \) and \( G \) are substitutes.

### III. The Impact of Depreciation

We can easily extend our analysis to include the possibility that public and private capital may depreciate at different rates. We are motivated in this extension by the casual observation that some types of governmentally supplied infrastructures (e.g., roads, port facilities, hydroelectric projects, etc.) seem to be more durable than those typically provided by private capital stock. Consider, therefore, replacing equations (3) with

\[
(3a') \quad P_t = (1 - \mu_p)P_{t-1} + \Delta P_{t-1},
\]

and

\[
(3b') \quad G_t = (1 - \mu_g)G_{t-1} + \Delta G_{t-1},
\]

where \( \mu_p \) and \( \mu_g \) represent depreciation rates for private and public capital, respectively. While these rates are presumed constant over time, we do preserve a measure of generality by allowing that \( \mu_p \) need not equal \( \mu_g \).

Following the analysis through, we can deduce that

\[
(29) \quad \Phi^*_G(t) = (r_t + \mu_g) + (\Phi_{PG}/\Phi_{PP})^*_t(k_t - (r_t + \mu_p)).
\]

If the rates of depreciation are equal, \( \Phi^*_G(t) \) reflects only the changes implied by the new public investment criteria. One small change is produced, however, when \( \mu_p \neq \mu_g \). Suppose, for example, that government capital does depreciate more slowly, i.e., suppose that \( \mu_g \) falls below a given \( \mu_p \). The optimal gross rate of return is then reduced regardless of the sign of \( \Delta_{PG} \). To the degree that public capital is more durable, its accumulation should therefore be encouraged. We should note, however, that we have captured only the obvious effect that increased durability simply increases the net return to capital for a given gross return. Rewriting (29), we see that
duplicates (12) when it is cast in terms of net returns. The conclusions that were drawn from the previous sections survive this complication entirely intact.

IV. CONCLUDING REMARKS

The government is, of course, typically faced with a multitude of potential projects whose services may be related by a complicated web of complementarities and substitutabilities. Each must therefore be appraised in the context of the interdependencies it creates with other existing and proposed projects. We can infer from our analysis, however, that it is equally important for the appraiser to take careful note of each project's complementarity or substitutability with privately provided capital. More specifically, when private investment is discouraged (encouraged) by distortions resulting from other policies, projects that would provide complementary services should be allowed lower (be required to achieve higher) rates of return than projects that would replace the private investments; the distortions are thereby partially undone. Regardless of these complications, durable projects should still be encouraged by allowing them lower gross returns. While developed countries provide public capital with varying degrees of interdependence and durability, these observations are perhaps most applicable to developing countries in which a large portion of public investment lies in the provision of infrastructures.

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