Single-valued control of a cartel under uncertainty—a multifirm comparison of prices and quantities

Gary W. Yohe

Department of Economics
State University of New York at Albany

Homogeneous and hybrid price and quantity controls of a cartel seeking to maximize cumulative profits are compared within an uncertain economic environment. The primary determinant of the superior control is shown to be the relative influence each choice has on the variation in total output. A member firm's size, relative to the total output, and the correlation of its output with the outputs of the other firms are therefore crucial in predicting whether the firm should optimally face a price or a quantity. Extensions of the analysis to pollution control, agricultural supports, and planned economies are also outlined.

1. Introduction

■ The recent literature has recorded several comparisons of single-valued price and quantity controls within an uncertain economic environment (see, e.g., Poole, 1969, Spence and Roberts, 1974, Weitzman, 1974, and Yohe, 1975). These studies undermine the traditional western view that, all things being equal, prices are always the better choice. It has now been demonstrated that there do indeed exist circumstances in which the *ex post* distributions of output created by the opposing choices imply that direct quantity regulation is socially preferable. Unilateral imposition of price controls is thus an inferior strategy.

A researcher who casually applies these general conclusions to a more structured problem, may, however, overlook the potentially significant impact of that structure on the comparison of prices and quantities he means to conduct. The present paper explores one such problem; we shall consider the simultaneous control of a cartel that is exercising the market power of a group of firms to maximize total profits. While a comparison of homogeneous, single-valued controls will be a primary concern, the possibility of mixing the control, so that some firms face a price specification at the same time others face quantities, will also be explored. We shall be able to demonstrate the conditions under which such a mix is superior to homogeneous regu-

This discussion is based on the third chapter of the author's Ph.D. dissertation submitted to Yale University in the Spring of 1975. The comments on earlier drafts by Professors William Brainard and J. Michael Montias of Yale and Martin Weitzman of M.I.T. are gratefully acknowledged, as is the encouragement offered by Gerome LeChat and my wife, Linda.

lation of either type. The effect of changes in the number of members, as well as the impact of a member's size, will also be noted.

The operation of a cartel is, of course, a natural and interesting framework within which to explore such questions. In addition, the terminology that emerges from this context will be of sufficient generality to allow straightforward extensions of the results to many other areas in which a group of productive units is to be regulated. A series of footnotes will facilitate such extension by recording corresponding interpretations of the primary conclusions in terms of social welfare. It should become clear to the thoughtful reader that a list of potential applications is both long and diverse; we have space to record but a few.

Pollution control comes to mind immediately. Regulation of all of the fixed source emitters of a given pollutant in a particular airshed provides a perfect example of a parallel comparative problem: should effluent changes be imposed in lieu of quantity standards to effect the requisite air quality? The purists will quickly point out that our study misses some important characteristics of the pollution problem. Pollutants are, for example, really best thought of as inputs of production for which a price will be paid, not received. The possibility of various degrees of substitutibility with other inputs must be considered.[1] A second difficulty lies with the consumptive character of a pollutant; the quantity consumed is related to the quantity produced by means of a plethora of weather-related random variables. This specialized type of uncertainty must also be analyzed. Even though we shall be ignoring such complications, our conclusions will contain an intuition about the control of multiple sources that would survive their inclusion.

Agricultural supports are a second obvious area of potential application. In the United States, the question becomes whether to institute price supports or acreage constraints in an effort to maintain the profitability of farming. That debate continues even now after long years of argument. The final entry of our brief list is, perhaps, the most obvious of all; within a planned economy, the regulation of the output of an entire sector is the fundamental problem. Shortages in a single sector can cause an entire plan to falter. The reader should note that such planning is not limited to the Soviet-type economies alone. Internal control of multiple suppliers can be equally important to a multiplant conglomerate embedded in a decentralized economy. As we turn now to the analysis, we do so with the certain knowledge that our conclusions will have far-reaching applicability.

2. The basic model

■ We have already stated that our cartel is interested in maximizing the cumulative profits of its membership. We assume that total revenues are derived from a downward sloping demand curve which, in turn, depends on total output, q, and a random variable, β. Revenues can therefore be represented by $R(q,\beta)$ such that

[1] The installation of an effluent cleansing device on a smokestack is an example of such substitution. When pollutants are viewed as inputs and appear in the production function, the elasticity of substitution reflects the ease with which such devices can be effectively employed.

$R_{11}(q,\beta) < 0$ for all (q,β).[2] The variable β is meant to reflect imprecise knowledge of the demand schedule, as well as desultory shocks to the schedule, itself. Each of the n members of the cartel is similarly working with an individual cost function that also exhibits a random variable, α_i, in its argument. These variables reflect day-to-day fluctuations in cost that can randomly beset any firm; they are not meant to reflect imprecise knowledge of costs at the firm level. We can summarize these notions by representing the cost function of the ith firm with $C^i(q_i, \alpha_i)$, where q_i is its output and $q \equiv \sum_{i=1}^{n} q_i$.

We assume, in addition, that

$$C_1{}^i(q_i, \alpha_i) > 0,$$
$$C_{11}{}^i(q_i, \alpha_i) > 0, \text{ and}$$
$$C_1{}^i(0, \alpha_i) < R_1(0, \beta),$$

for all (q_i, α_i) and (α_i, β), and finally that β and the α_i are jointly distributed.

So that our attention can be focused on the relative merits of price and quantity controls, the governing body of the cartel (henceforth, the governor) is constrained to the issuance of a single-valued, once and for all order to each firm. If these orders must be made before the true values of the random variables are known, the governor must select the optimal specifications by maximizing expected revenue minus expected costs. We should note that while we shall be equipping the governor with precise knowledge of the distribution function, our substantive results are also equally valid when the center operates with an incorrect perception of that density (see Yohe, 1975). In either case, our comparison of the two potential modes of control is based on an accurate computation of expected profits. The underlying responses of the various members thus become crucial. For the moment, we assume that a member will respond precisely to a quantity order, regardless of the cost. In response to a price order, on the other hand, that same firm is presumed to maximize its own profits by setting actual marginal cost equal to the given price. Our comparison will reflect this inherent asymmetry.

One final technicality will make the mathematics more tractible. The assumed shapes of the revenue and cost functions guarantee the existence of a set of positive qualities $\{\hat{q}_i\}_1{}^n$, such that

$$E\{R_1(\Sigma\hat{q}_i, \beta)\} = E\{C_1{}^i(\hat{q}_i, \alpha_i)\} \tag{1}$$

for all i. We shall use these $\hat{q}_i$ to construct quadratic representative functions of the following form:

$$R(q,\beta) \equiv (R'+\beta)(q-\hat{q}) + (R''/2)(q-\hat{q})^2, \text{ and} \tag{2a}$$

$$C^i(q_i,\alpha_i) \equiv (C'_i + \alpha_i)(q_i-\hat{q}_i) + (C''_i/2)(q_i-\hat{q}_i)^2 \tag{2b}$$

for all i and where $\hat{q} \equiv \Sigma\hat{q}_i$.[3] Without loss of further generality, we can

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[2] We could just as easily use $R(q,\beta)$ to represent a benefit schedule and cast the analysis in terms of a central regulatory agency seeking to maximize social welfare. This context would formalize the secondary applications listed in the introduction.

[3] The reader will note that equations (2) look like second-order Taylor series approximations of arbitrary revenue and cost functions. Indeed, the model captures more generality if we admit to these origins by assuming that $R(\hat{q}, \beta) = C^i(\hat{q}_i, \alpha_i) = 0$, for all i, and defining $R_1(\hat{q}, \beta) \equiv (R' + \beta)$, $R_{11}(\hat{q}, \beta) \equiv R_{11}$, $C_1{}^i(\hat{q}_i, \alpha_i) \equiv (C_1{}^i + \alpha_i)$, and

interpret R' and the C'_i as the means of the random disturbances affecting benefits and costs, respectively. Since these disturbances have been confined by assumption to the intercepts of the various marginal schedules, their total impact is captured by β and the α_i. This interpretation also allows us to observe that $E\beta = E\alpha_i = 0$, for all i, so that $R' = C'_i \equiv C'$ is guaranteed by equation (1). We can therefore finally record that

$$C^i(q_i, \alpha_i) = (C' + \alpha_i)(q_i - \hat{q}_i) + (C''_i/2)(q_i - \hat{q}_i)^2 \tag{2b'}$$

for all i.

3. Homogeneous control of the cartel

■ It is clear from our definition of the $\{\hat{q}_i\}$ that these outputs are the optimal, single-valued quantity controls for the n members. Computing the corresponding optimal price order is, however, considerably more involved. We require, for any specified price, that each firm read the value of its particular random vector of cost variables, α_i, and maximize actual profits; that is to say, each firm will produce $q_i(p, \alpha_i)$ such that

$$p = C' + \alpha_i + C''_i(q_i(p, \alpha_i) - \hat{q}_i).$$

Each firm's price reaction schedule therefore becomes

$$q_i(p, \alpha_i) = \hat{q}_i - (C' + \alpha_i - p)/C''_i,$$

and the governor must maximize

$$E\left\{R(q_i(p, \alpha_i), \beta) - \left[\sum_{i=1}^{n} C^i(q_i(p, \alpha_i), \alpha_i)\right]\right\}$$

with respect to p to determine the best command, $\tilde{p}$. Since $(\partial q_i/\partial p)$ is nonstochastic, the center's first-order condition reads

$$\tilde{p} = R' + R'' \left(\Sigma(\tilde{p} - C')/C''_i\right),$$

and $\tilde{p} = R' = C'$. The quantity reaction to the optimal price for any firm is finally given by

$$\tilde{q}_i(\alpha_i) = \hat{q}_i - (\alpha_i/C''_i). \tag{3}$$

□ **The comparative advantages of prices.** We are now able to compare the relative merits of these rival modes of homogeneous control by computing the difference between the level of expected profits achieved under price control by $\tilde{p}$ and the corresponding level achieved by quantity control by the $\{\hat{q}_i\}$:

$$\Delta_n = E\{[R(\Sigma\tilde{q}_i(\alpha_i), \beta) - \Sigma C^i(\tilde{q}_i(\alpha_i), \alpha_i)]$$
$$-[R(\Sigma\hat{q}_i, \beta) - \Sigma C^i(\hat{q}_i, \alpha_i)]\}. \tag{4}$$

This statistic is, of course, the multifirm profit analog of Weitzman's comparative advantage of prices over quantities (see Weitzman, 1974). When Δ_n is positive, unilateral price control of the cartel is preferred; when Δ_n is negative, quantities are favored.

$C_{11}{}^i(\hat{q}_i, \alpha_i) \equiv C_{11}{}^i$. If F is compact and the variances of the random variables are small, then such approximations are entirely acceptable (see Samuelson, 1970). The skeptical reader is referred to Yohe (1975) for a demonstration that continuing to higher terms reveals little in the way of additional economic insight.

It is easily demonstrated, by inserting (3) into (4), that

$$\Delta_n = \sum_{i=1}^{n} \sum_{j=1}^{n} (R''/2)E\{\alpha_i\alpha_j/C''_iC''_j\}$$

$$+ \sum_{i=1}^{n} (C''_i/2)E\{\alpha_i/C''_i\}^2$$

$$- \sum_{i=1}^{n} E\{\beta\alpha_i/C''_i\}.$$

When we further recall that $E\alpha_i = 0$ for all i, we can observe, in addition, that Δ_n can be reduced to a form that is more easily interpreted:

$$\Delta_n = (R''/2)\mathrm{Var}(\Sigma\tilde{q}_i(\alpha_i)) + (\Sigma(C''_i/2)\mathrm{Var}(\tilde{q}_i(\alpha_i))$$

$$+ \mathrm{Cov}(\Sigma\tilde{q}_i(\alpha_i)\ ;\ \beta). \tag{5}$$

The first term reflects the pure effect on the revenue side of price-induced variation in total output. Such variation causes expected revenues to fall below the level that would be achieved were $\hat{q}$ produced with certainty. This loss, of course, increases with both the variance of output and the absolute magnitude of R'' (and thus with the price elasticity of demand). The last term of equation (5) also records an effect of variation in total output under prices, now viewed in the context of a randomly shifting marginal revenue schedule. When that covariance is positive [negative], total output tends to increase [decrease], just when marginal revenue shifts upwards. This being the correct [wrong] direction, a positive bias for prices [a negative bias against prices] is recorded. Only the middle term registers disaggregated effects that occur at the individual member firms. At each firm, output variation under prices causes expected costs to rise above the level that would be incurred were $\hat{q}_i$ produced with certainty—a loss registered by the expression $[-(C''_i/2)\mathrm{Var}(\tilde{q}_i(\alpha_i))]$ for all i. In addition, because marginal costs are equal to $\tilde{p}$ across the industry, output at each firm always moves in the correct direction relative to actual costs—an efficiency gain registered by $\mathrm{Cov}(\alpha_i;\ \tilde{q}_i(\alpha_i)) = C''_i\ \mathrm{Var}(\tilde{q}_i(\alpha_i))$ for all i. The net gain for each firm is therefore $(C''_i/2)\ \mathrm{Var}\ (\tilde{q}_i(\alpha_i))$, and the middle term in (5) merely sums these gains over the entire industry. The choice between price and quantity controls therefore comes down to comparing these various effects of output variation in the context of curved, randomly shifting cost and total revenue schedules.[4]

While equation (5) is an expositionally convenient expression for the comparative advantage, it does submerge a potentially important diversification effect that could be exploited in the multifirm case. A rewrite of (5) exposes that potentiality in its second term:

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[4] Were we viewing R as a benefit function, the interpretation is changed only slightly. Variation in total output would diminish expected benefits to an extent determined by the curvature of the benefit function and the size of the variation. The correlation effects can also be cast perfectly well in terms of a randomly shifting marginal benefit schedule; e.g., an increase in output when marginal benefits shift up is still the correct direction.

$$\Delta_n = 1/2\sum_{i=1}^{n}(R'' + C''_i)\mathrm{Var}[\tilde{q}_i(\alpha_i)] + \sum_{i=1}^{n}\sum_{j=i}^{n} R''\mathrm{Cov}[\tilde{q}_i(\alpha_i);\ \tilde{q}_j(\alpha_j)] + \mathrm{Cov}[\beta;\ \Sigma q_i(\alpha_i)].$$

For the sake of descriptive clarity, we shall compare the case in which firms' cost schedules are all independent with a case in which only one covariance (between firms 1 and 2) is nonzero. The price induced output variations of these two firms will tend to be in the same direction whenever $\mathrm{Cov}(\tilde{q}_1;\ \tilde{q}_2) > 0$. Taken together, changes in the individual firms' outputs will then amplify each other, and increase the variance of total output over the case of complete independence. Any such increase is detrimental, and an additional bias against prices $[R''\mathrm{Cov}(\tilde{q}_i;\ \tilde{q}_2) < 0]$ is recorded. When the covariance is negative, however, the relevant outputs move in opposite directions and tend to cancel. Total output variation is reduced, and a bias for prices noted. We have seen, therefore, that prices become more preferred (or quantities less) as the cost schedules of the individual firms become more negatively correlated, because the potential gains from diversification are increased.

□ **Changing the number of members.** In tracing the impact of altering the number of firms, we shall be interested in capturing only the pure effect of such a change, and not the influence of some secondary cost or production changes that might distort the character of the individual firms. It will also help us to concentrate our analytic attention if we assume n identical firms that are identically distributed; in that case,

$$\begin{gathered} C''_i = C'', \\ \mathrm{Var}\,[\alpha_i/C''] = \mathrm{Var}[\tilde{q}_i(\alpha_i)] = \sigma^2, \text{ and} \\ \mathrm{Cov}\,[\tilde{q}_i(\alpha_i);\ \beta] = \omega, \text{ for all } i, \text{ while} \\ \mathrm{Cov}\,[\tilde{q}_i(\alpha_i);\ \tilde{q}_j(a_j)] = \rho\sigma^2, \text{ for all } i \neq j. \end{gathered}$$

We can subsequently avoid the first difficulty by working with a transformed cost function:

$$\Gamma(q,\ \alpha_i) \equiv n\ C(q/n,\ \alpha_i).$$

The function Γ can be thought to relate a given *total* output to *total* cost, under the assumption that all firms are identical. We can observe further $\Gamma_1(q,\ \alpha_i) = C_1(q_i,\ \alpha_i)$, $\Gamma_{12}(q,\ \alpha_i) = C_{12}(q_i,\ \alpha_i)$, and $\Gamma_{11}(q,\ \alpha_i) = [C_{11}(q_i,\ \alpha_i)]/n$. As a result, the variance-covariance matrix of marginal costs survives the transformation completely intact. The corresponding matrix of output variation under price control is similarly preserved. In considering a *ceteris paribus* change in n, we shall be holding the revenue function and the transformed cost function fixed. The preceding observation verifies that we shall be able to hold ω, ρ, and σ^2 constant, as well.

The comparative advantage of prices emerges from these restrictions and our quadratic formulae in the following form:[5]

$$\Delta'_n = \rho(R''\sigma^2/2 + \Gamma''\sigma^2/2) + (1-\rho)(R''\sigma^2/2n + \Gamma''\sigma^2/2) + \omega.$$

[5] By applying our restriction of identical firms to the transformed cost function, we can observe that $\Gamma_{11}(q,\ \alpha_i) = (C''/n) \equiv \Gamma''$.

Our first result is then immediate.

Proposition 1: A *ceteris paribus* increase in the number of firms contributing to the total output creates a corresponding increase in the comparative advantage of industry-wide price control over industry-wide quantity control, given any finite degree of interfirm independence.

Before we explore the economic forces that underlie this result, a few remarks are in order. Notice initially that a large n does not guarantee a positive comparative advantage:

$$\lim_{n\to\infty} \Delta'_n = \frac{\rho R''\sigma^2 + \Gamma''\sigma^2}{2} + \omega \gtrless 0.$$

In addition, our result in no way depends on our condition that all firms be identical. Insertion of transformed cost functions into (5) would lead to the same conclusion by means of more arduous reasoning.

Turning to the economic genesis of the proposition, the first influence that comes to mind is the *ex post* efficiency gain afforded price control by their guarantee that marginal cost be equal across firms; that is, $C_1(\tilde{q}_i(\alpha_i),\ \alpha_i) = C_1(\tilde{q}_j(\alpha_j),\ \alpha_j) = \tilde{p}$, for all i and j. Price controls therefore automatically screen high cost producers and encourage them to produce less. Low cost firms are similarly encouraged to produce more. Under quantity control, on the other hand, $C_1(\hat{q}_i,\ \alpha_i) \neq C_1(\hat{q}_j,\ \alpha_j)$, since $i \neq j$ except in very special circumstances of zero probability. One such case occurs when the $\{\alpha_i\}$ are perfectly correlated ($\rho = 1$); marginal costs would then be equal across firms regardless of the control choice, and the efficiency gain disappears. Proposition 1 therefore requires at least a finite degree of interfirm independence. Indeed,

$$\Delta'_n(\rho = 1) = (R'' + \Gamma'')\sigma^2/2$$

is independent of n, and simply represents the comparative advantage of prices under the assumption that the cartel is a single firm.

A second influence can be uncovered if we compute a set of points $\{q'_i\}$ such that

$$\sum_{i=1}^{n} \hat{q}_i = \sum_{i=1}^{n} q'_i, \text{ and}$$

$$C_1(q'_i,\ \alpha_i) = C_1(q'_j,\ \alpha_j),$$

for $i \neq j$; that is, the $\{q'_i\}$ are selected not only to equalize marginal costs across firms, but also to maintain total output at the level prescribed by quantity controls. Solving for the $\{q'_i\}$ by using Cramer's Rule and induction on the number of firms, we find that

$$q'_i = \hat{q}_i - [(n-1)\ \alpha_i - \sum_{\substack{k=1 \\ k\neq i}}^{n} \alpha_k]/nC'', \tag{6}$$

for all i.

We observe, first of all, that if the $\{\alpha_i\}$ are perfectly correlated, then $\alpha_i = \alpha_j$ for all i and j, and $\hat{q}_i = q'_i$. As we have already noted, prices do not collect an efficiency gain because marginal costs are equal even under quantity control. On the other extreme, when the $\{\alpha_i\}$ are independently distributed,

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} (\alpha_k) \simeq (n - 1)E(\alpha) = 0$$

as n grows large. As a result,

$$\lim_{n \to \infty} q'_i = \tilde{q}_i(\alpha_i), \text{ for all } i;$$

in the limit, therefore, we can compute output levels for all of the firms so that their marginal costs are precisely equal to the optimal price order without moving *total* output from the level specified by optimal quantity control. While most cases lie between these extremes, the intuition they provide can certainly be extrapolated. Price controls garner a gain from diversification that lies in the ability, as n increases, to set marginal costs of each firm closer to the others' without altering total output. Put another way, the total output variation that results from setting marginal costs at each firm equal to $\tilde{p}$ becomes monotonically smaller as the number of firms increases. Since it is output variation that hurts prices, the diminishing of that variation is a positive bias in their comparative advantage.

4. Policy mixes with an industry

■ We have, thus far, restricted our consideration to the homogeneous regulation of the entire cartel. It is, however, quite possible for uniform control by either mode to be inferior to a mixed strategy in which some of the members face prices while others face quantities. In the present section, therefore, we fully explore this possibility, and ultimately derive conditions for the existence of a superior mix. The reader will shortly observe that the presumption of identical firms has lapsed.

□ **Optimal control orders under a mixed scheme.** We begin our investigation of mixed strategies by computing the control orders that would constitute an arbitrary mix. Consider, to that end, a potential mix in which the first m firms maximize profits in the face of a price order, while the remaining $(n - m)$ operate under direct quantity regulation. The governor must therefore maximize

$$E\left\{R\left[\left(\sum_{i=1}^{m} q_i(p, \alpha_i) + \sum_{i=m+1}^{n} q_i\right), \beta\right] - \sum_{i=1}^{m} C^i(q_i(p, \alpha_i), \alpha_i) - \sum_{i=m+1}^{n} C^i(q_i, \alpha_i)\right\}$$

with respect to p and q_{m+1} through q_n. We can see immediately from the last $(n-m)$ first-order conditions that the optimal quantity orders remain $\{\hat{q}_i\}_{m+1}^{n}$. The remaining first-order condition requires, in addition, that the new price, $\tilde{p}'$, satisfy

$$\tilde{p}' = R' + R''\left(\sum_{i=1}^{m} (\tilde{p}' - C')/C_i''\right).$$

By further noting that $B' = C'$, we conclude finally that the price order also survives the mix intact; i.e., $\tilde{p}' = \tilde{p}$.

Since both the optimal price and the price reaction schedules for the first m firms are the same as before, the quantity responses to $\tilde{p}$ are similarly identical; i.e.,

$$\tilde{q}_i(\alpha_i) = \hat{q}_i - (\alpha_i/C_i'')$$

will be produced by each. We have demonstrated, therefore, that the quantity decisions made at either seat of authority under a mixed policy are precisely those that would have been made under the corresponding homogeneous control.

☐ **General conditions for a superior mix.** We can attain maximum generality by contrasting our arbitrary mix, now designated mix I, with two alternatives that differ only in the treatment of a single firm. The first, mix II, duplicates I but for the $(m+1)$st firm; that firm is switched to price control. The comparative advantage of mix II over mix I is then simply the difference in the levels of expected profits that they achieve. Algebraic manipulation subsequently reveals that

$$\Delta(\text{II/I}) = [C''_{m+1}\text{Var}(\tilde{q}_{m+1})]/2 + [R''\text{Var}(\tilde{q}_{m+1})]/2 + R''\sum_{i=1}^{m}\text{Cov}(\tilde{q}_{m+1};\, \tilde{q}_i) + \text{Cov}(\tilde{q}_{m+1};\, \beta). \tag{7}$$

The various terms of equation (7) are easily rationalized; we have seen them all before. The first represents the positive net bias of the efficiency gain that is achieved under mix II by guaranteeing that the $(m+1)$st firm sets actual marginal costs equal to $\tilde{p}$. The dampening effect of output variation on expected costs is, as usual, included. The middle two terms similarly represent the net loss in expected revenues that is created because switching the control of the $(m+1)$st firm changes the variation pattern of total output; the basic loss represented by $(R''/2)\text{Var}(\tilde{q}_{m+1})$ is amplified (or dampened) by positive (or negative) correlations with the outputs of the other m firms already regulated by prices. The single remaining term is also familiar, reflecting the previously explained correlation effect between the varying output of the switching firm and the randomly shifting marginal revenue schedule.

In dissecting (7), however, the reader may have overlooked a more significant interpretation. The comparative advantage of mix II over mix I is precisely equal to the comparative advantage of prices over quantities for the $(m+1)$st firm, if it is considered individually in the context of its position within the cartel as otherwise regulated by the original mix. That position can be seen to be dependent on two factors. The first is defined by the array of interdependencies between its output, the outputs of the other firms, and the random elements of the marginal revenue schedule that we have just described. The joint distribution of the random variables, of course, creates these interdependencies. The second crucial factor is the size of the $(m+1)$st firm relative to the rest of the cartel. We can illustrate this factor by rewriting $\Delta(\text{II/I})$ in terms of the transformed cost function defined above in Section 3:

$$\Delta(\text{II/I}) = (1/n)\{[\Gamma''_{m+1}\ \text{Var}(\tilde{q}_{m+1}) + (R''/n)\ \text{Var}(\tilde{q}_{m+1})]/2 + \left[(R''/n)\sum_{i=1}^{m}\text{Cov}(\tilde{q}_{m+1};\, \tilde{q}_i)\right] + \text{Cov}(\tilde{q}_{m+1};\, \beta)\}. \tag{7a}$$

Observe that the entire expression is multiplied by $(1/n)$; when the membership is large, switching the control of a single firm should have little effect on overall welfare. In addition, and consistent with Proposition 1, we can also observe that an increase in the number of firms increases the likelihood that a shift to price control of the marginal firm will be profitable.

As should be expected, reversing the switch produces a very similar result. Consider a third mix which differs from I only at the mth firm that is to be switched to quantities. The comparative advantage of mix III over mix I is then given by

$$\Delta(\text{III/I}) = -(1/n)\Big\{(1/2)[\Gamma''_m \operatorname{Var}(\tilde{q}_m) + (R''/n) \operatorname{Var}(\tilde{q}_m)] + (R''/n)\Big[\sum_{i=1}^{m-1} \operatorname{Cov}(\tilde{q}_m; \tilde{q}_i) + \operatorname{Cov}(\tilde{q}_m; \beta)\Big]\Big\}. \tag{8}$$

The only substantive change is the minus sign that converts losses into gains and gains into losses; but (III/I) reflects a comparative advantage of quantities over prices, so this transformation is to be expected. The content of (8) is otherwise identical to (7a), and we have demonstrated the following proposition.

Proposition 2: In order to guarantee the profitability of an alteration in the control mix, it is sufficient to show the existence of one firm for which the opposite mode of control is preferred when that firm is considered individually, but in the context of the industry as otherwise controlled by the original mix.

Several remarks are in order before we close this section. First of all, by setting m equal to either n or zero, we have sufficient conditions for the existence of a mix that is preferred to uniform control by either prices or quantities, respectively. In addition, even though the existence of an optimal mix can be insured by checking the conditions of Proposition 2,[6] simply changing the mode of control of all firms that so prefer on an individual basis need not yield a global maximum. It is quite possible, for instance that making a switch in control with a negative comparative advantage could set up a covariance structure that would ultimately attain a higher value of expected profits.

To see this point, we begin with a mix in which the first m firms are regulated by quantities ($m \geqslant 2$), while the remaining $(n - m)$ firms face prices. Suppose, in addition, that although there do not exist firms for which a change of control would be preferred on an individual basis, there does exist a subgroup of t firms (numbered $(m - t + 1)$ to m) under quantity control for which prices would be favored if they were considered together. If we can show that these two conditions can be met simultaneously, we will have illustrated our *caveat*. Notice, too, that we will have proven that the sufficient conditions of Proposition 2 are not necessary. Our first assumption requires that

$$\Big\{C_k'' \operatorname{Var}(\tilde{q}_k) + R''\Big[\operatorname{Var}(\tilde{q}_k) + 2\sum_{j=m+1}^{n} \operatorname{Cov}(\tilde{q}_k; \tilde{q}_j)\Big]\Big\}/2 + \operatorname{Cov}(\tilde{q}_k, \beta) \equiv G(k) < 0$$

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[6] We have a finite number of possible mixes. Their respective levels of expected profits constitute a closed and bounded subset of the real line. That subset must therefore contain its own maximal element.

for $k = (m-t+1), \ldots, m$. Our second assumption similarly requires that

$$\sum_{k=m-t+1}^{m} \left\{G(k) + (1/2)R'' \left[\sum_{\substack{j=m-t+1 \\ j \neq k}}^{m} \mathrm{Cov}(\tilde{q}_k; \tilde{q}_j)\right]\right\} > 0. \qquad (9)$$

It is quite possible for $G(k)$ to be negative for all k and for the total value of (9) to be positive. All that is required is a strong negative correlation of cost within the given subgroups of firms. We should therefore not be lulled into looking at only one firm subgroups by the dependence of our proposition on simple single firm conditions.

5. Large firms and a preferred mix

■ When we envision a cartel, we typically think of a collection of large and small firms producing the same output. It is therefore reasonable to ask what influence the relative size of a firm exerts on the choice of control in a preferred mix. We propose two methods of introducing a large firm into the current analysis, in response to that query. Each method involves viewing such a firm as a collection of highly correlated production units. On the one hand, a single cost function for a large firm can be determined by the horizontal addition of the cost curves of many smaller production units; we thereby create a cost curve with a smaller curvature than any of the single units. In the context of this notion, then, we suggest representation of a large firm by a cost function with a small value for C''. On the other hand, we can preserve the individual production units by defining a large firm to be a collection of perfectly correlated production units with values of C'' more in line with the small firms.

Under either interpretation, we can now show that large firms are more likely to be regulated by quantities than prices in a preferred control mix. If we view such a firm as a collection of perfectly correlated units that must face the same control, for instance, the comparative advantage of prices for that firm in the context of an otherwise arbitrary mix is again given by equation (9).[7] Since we require that the units within the firm be perfectly correlated, we observe that (9) has a large negative term in its very heart. This negative bias wold be extremely difficult to overcome unless the production units either faced extremely high curvatures in their cost functions (C_i''), or were extremely negatively correlated with the other firms under price control. There is no compelling reason to believe that either condition is very likely, and it should be expected that large firms face quantity regulation.

The alternative is to view a large firm as a single production unit (k) with a very small cost curvature (C_k''). The comparative advantage of prices in the context of an arbitrary mix is then simply $G(k)$. Because a small curvature implies a large output variation under prices, $G(k)$ is also shackled with a negative bias that can be over-

[7] The precise specification of the case that generates this comparative advantage is the following: a cartel of n production units, p of which form our large firm. We have numbered the units so that the first $(m - p)$ face $\tilde{p}$, the last $(n - m)$ face quantities, and the middle p are the subject of discussion.

come only by negative cost covariances with the other firms under price controls. As we have already noted, there is no *a priori* reason to predict such correlation, and quantity control emerges more likely. We are reminded in closing, however, that the large negative bias that is observed in either representation is not sufficient to preclude a preferred mix in which a large firm faces price control.

6. Output disturbance under quantity control

■ We have thus far assumed that *ex ante* quantity orders are met with absolute certainty by every firm, while the corresponding price orders produce varying levels of output that depend upon the *ex post* state of nature. The astute reader may fear that such asymmetry creates a systematic bias for quantity controls. It is, after all, output variation that causes the level of expected profits to change. In this final substantive section, therefore, we shall record the results of an investigation of that concern, and provide intuitive explanations of their origins.

The simplest way to correct for the troublesome asymmetry was to assign a second random variable to each firm that could cause its response to a quantity order to deviate from the prescribed level in either direction. Since we should expect these variables to influence costs as well as output, they were also inserted into the stochastic elements of the individual cost functions. They therefore had an indirect influence on the output responses to a price order; that effect paralleled the impact of the α_i outlined above, and the optimal price order thus remained C'. Their direct effect on output under quantity control was not neutral, however, and the center was observed changing the quantity order in such a way that the expected production levels were precisely equal to the previously selected $\hat{q}_i$.

This last observation made the comparative advantage of homogeneous price controls easy to interpret; the two types of control differed only in the character of the output variation that they allowed around identical means. The price related terms recorded in equation (5) persisted to reflect the same losses and gains as before. Variation under quantities meanwhile produced similar terms that indicated the corresponding losses and gains under quantities. Variation under quantity controls, for instance, caused expected revenues to fall below the level achieved when $(\Sigma \hat{q}_i)$ was produced with certainty. The magnitude of this loss depended on the curvature of the revenue function, as well as the variance in total output, and it biased the comparison against quantities. Total output variation also appeared in conjunction with the simultaneous variation of marginal benefits. If total output under quantity control tended to increase at the same time that marginal benefits were high, for example, a bias toward quantities would be felt and recorded in the comparative advantage.

The introduction of uncertain output under quantity control also produced familiar repercussions on the cost side; there was only one significant difference. While the expected costs of each firm rose above the certainty levels, the counterbalancing efficiency gains that guaranteed a positive cost side for prices did not appear in support of quantities. The correlation effects between the output of the individual firms and their marginal cost schedules had to be computed separately; when they were summed to record the efficiency impact across the entire cartel, their total effect was ambiguous.

The case for quantity controls was therefore seriously weakened, but there still existed many situations in which they were to be preferred over prices. Uncertain responses to quantity standards were seen, in general, to make the choice depend on the relative magnitudes of the output variances created by the two alternatives. The profitability of mixed controls was also affected, but Proposition 2 remained valid. A new set of quantity-spawned output variations was observed to have altered a member's position in the cartel. As we contemplated changing the control of the *i*th firm from prices to quantities, we had previously considered only the correlations between the *i*th firm's output and the outputs of the other price regulated firms. Those correlations disappeared in the switch. In this second formulation, it was necessary to deduct output correlations between the *i*th firm under prices and the quantity regulated firms in addition to incorporating output correlations between the *i*th firm under quantities and all the rest as otherwise controlled. We were, however, still looking at an individualized comparative advantage.

Proposition 1 did not fare so well, since an increase in the number of members no longer unambiguously favored homogeneous price controls. While prices still stood to capture efficiency and diversification gains as n increased, quantity controls also exhibited a potential gain from diversification. Given at least a finite degree of interfirm independence, the type of control displaying the largest potential gain from these sources was found to be favored by an increase in the cartel's membership.

7. Concluding remarks

■ Extension of the one product case to include multiple producers reaffirms the strong influence to output variation on the prices-quantities comparison. The crucial determinant in the n-firm case, based on either profits or social welfare, is the variance in the total output of the cartel (industry). Variation in the output of each firm can influence only a fraction of this total output, and that influence is either amplified or dampened by simultaneous variations in the outputs of the other firms. Having thereby taken into account a firm's place in the cartel, we have shown that it is profitable to regulate that firm by the mode that would be preferred if it were considered individually, in the context of that position. As the number of firms increases, *ceteris paribus*, price controls are afforded both an efficiency gain and a diversification gain; quantity controls receive only a diversification gain. The relative magnitudes of these opposing gains then determine which mode receives a positive bias from an enlarged membership.

Direct application of these results to pollution control requires an assumption that the negatively valued pollutant appear with the positively valued output in fixed proportions; only then can a benefit function accurately summarize the influence of both products on social welfare. This is obviously a very restrictive condition, but we are still able to draw some general conclusions from the preceding analysis.[8] We need only observe that either a set of effluent charges

[8] Allowing the type of substitution suggested in the first footnote can be shown to influence only the importance of the benefit side of the comparison (see Yohe, 1976 and

or a corresponding set of quantity standards will produce, at best, a distribution of outcomes that depends on the random cost variables.

Suppose, for instance, that the targeted level of total emissions lies near the critical level of the dosage response curve for the pollutant in question.[9] The variances in total emissions under charges and standards would then be crucial; that is, the benefit schedule would be highly curved and such variation highly deleterious. The control, or mix, with the lowest total variance should thus be imposed. As the number of emitters increases, the likelihood of a preferred mix would also increase, and the crucial role would be played by the largest firms. The regulating agency should make quite certain that those large emitters face the correct control.

If, on the other hand, the target for total emissions were far from the critical level, the benefit schedule might be expected to be nearly linear in the relevant range. The cost side would then be dominant, and the efficiency gains afforded effluent charges by their self-screening impact suggest a positive comparative advantage of prices. The influence of the number of emitters is also diminished, in this case, since the cost side is independent of that number.

As we turn now to discuss agricultural supports, we find that the largest uncertainties, those created by the weather after the crops are planted, have a neutral impact on our comparison. Neither the regulatory agency nor the farmers are able to observe the weather *ex ante*, and they both act in accordance with their expectations. As long as these expectations coincide, no effect is observed. We can suggest, however, that a farmer's response to an acreage constraint will be different from his response to a price support. Since there is a biological limit to the amount of corn that can be grown on an acre of land, we should expect that a price support would generate a larger, more correlated variation in the total amount harvested than the corresponding acreage constraint. This is then a secondary uncertainty with which the center must deal in selecting its mode of control. Our results predict that if our suggestion is correct, an inelastic demand for the crop would imply the superiority of an acreage specification, especially for the largest farms and coops. Extensive empirical work is obviously required to extend this paragraph from a state of pure conjecture, but our analysis has revealed the crucial parameters.

Having recorded these results, it may be useful to close by responding to two potential lines of criticism in advance. For one thing, the informational difficulties that can beset a production hierarchy have been ignored entirely; in terms of our model, the governor may be forced to make control decisions without the benefit of accurate information. We can, however, view this problem in terms of the governor's working with an incorrectly specified subjective distribution of the relevant random variables. In that case, our conclusions emerge with but one small qualification: the variances and covariances displayed in the various comparative advantages must be

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1977). The intuition developed here concerning variation in the amount of total emissions survives intact.

[9] The dosage response curve traces the health impact of pollution as the concentration increases. It is generally believed that there is a critical level for each pollutant at which the curve begins to turn sharply upward as the concentration becomes intolerable.

measured around the center's inaccurate view of the relevant means (see Yohe, 1975).

The resulting suboptimal controls are, however, symptoms of a larger problem: the governor must devise an efficient system with which to collect the requisite information. While both types of control require the same information to be properly specified, inaccurate information will affect them differently. If, for instance, the governor relies on the peripheral membership to supply the cost information, he might receive figures that are systematically overstated.[10] This exaggeration will result in quantity orders that are too low, and price orders that are too high; the latter subsequently generates output levels that are too high, as well. In choosing an informational scheme, the losses that are derived directly from misspecified data in this way must be weighed against the costs of independent collection of policing the periphery.

A second worry may develop around our use of quadratic cost and revenue functions. Extensive analysis of third-order effects has, however, generated a geometric interpretation of their impact that is easily applied in this multifirm case. Output variation, of course, remains the crucial determinant, and its importance still depends on the curvatures of the benefit and cost schedules. A third-order term simply reflects how that curvature changes in response to a change in output. With this in mind, a researcher can not only deduce the impact of third-order effects, but also observe when that impact is likely to be significant.

References

HAYEK, F. A. "The Use of Knowledge in Society." *The American Economic Review,* Vol. 25, No. 4 (September 1935), pp. 519–530.

HEAL, G. M. "Planning Without Prices." *Review of Economic Studies,* Vol. 36 (July 1969), pp. 347–362.

MARGLIN, S. A. "Information in Price and Command Systems Of Plannings," in J. Margolis, ed., *Activity Analysis in the Theory of Growth and Planning,* New York: Macmillan, Inc., 1967, pp. 54–77.

POOLE, W. "Optimal Choice of Monetary Policy Instruments in a Simple Stochastic Macro Model." *Special Studies Papers,* No. 2, Federal Reserve Board, 1969.

SAMUELSON, P. A. "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances, and Higher Moments." *Review of Economic Studies,* Vol. 37 (October 1970), pp. 537–542.

SPENCE, A. M. AND ROBERTS, M. "Effluent Charges and Licenses under Uncertainty." *Stanford University Technical Report,* No. 146, 1974.

WEITZMAN, M. "Prices vs. Quantities." *Review of Economic Studies,* Vol. 41 (October 1974), pp. 50–65.

YOHE, G. "*A Comparison of Price Controls and Quantity Controls under Uncertainty.*" Unpublished Ph.D. dissertation, Yale University, 1975.

———. "Substitution and the Control of Pollution." *Journal of Environmental Economics and Management* (1976).

———. "Single-valued Control of an Intermediate Good Under Uncertainty." *International Economic Review* (February 1977).

[10] The motive behind such exaggeration is clear when the periphery is accountable for the costs it incurs. Each firm would be given an easier standard against which to perform if the governor were to believe that the exaggerated figures he received were accurate.