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CONSTANT ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTIONS WITH THREE OR MORE INPUTS
An Approximation Procedure

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An approximation procedure is developed to allow arbitrary and constant elasticities of substitution between aggregate inputs in production functions with three or more factors. The procedure is necessary to overcome the serious constraints on structure imposed on CES schedules by the Uzawa–McFadden results.

1. Introduction

In 1962 H. Uzawa published a result that seriously diminished the applicability of CES production schedules in cases for which more than two inputs must be considered. It states that if the elasticities of substitution between every pair of inputs are to be held constant, then one of the following two conditions must be satisfied:

(1) the elasticities of substitution between all input pairs must be identical, or
(2) the elasticity between at least one pair of inputs must be equal to -1.

Table 1 catalogs these conditions in the three factor case to demonstrate clearly the restrictions that the theorem imposes on a scholar who wants to employ CES schedules in applied research: to the extent that the application may not conform to any of these structures, the relative simplicity of parameterizing substitution with a single number would appear to be lost.

In some circumstances, of course, the loss of this simplicity is merely

Table 1
Conditions of the Uzawa theorem on the three factor case.

Condition	Representation
(1) Identical elasticities	$[\alpha_1 x_1^\delta + \alpha_2 x_2^\delta + (1 - x_1 - \alpha_2) x_3^\delta]^{1/\delta}$
(2) One pair elasticity unity	$x_1^\alpha [\beta_2 x_2^\delta + (1 - \beta_2) x_3^\delta]^{(1-\alpha)/\delta}$ $[\beta_1 (x_1^\alpha x_2^{1-\alpha})^\delta + (1 - \beta_1) x_3^\delta]^{1/\delta}$

Elasticities	
$\sigma_{12} = \sigma_{13} = \sigma_{23} = (\delta - 1)^{-1}$	
$\sigma_{12} = \sigma_{13} = -1$	and $\sigma_{23} = (\delta - 1)^{-1}$
$\sigma_{12} = -1$	and $\sigma_{13} = \sigma_{23} = (\delta - 1)^{-1}$

the inevitable cost of looking at a complicated economic system. In others, however, the cost may be more severe. Even though reality may not conform well to the constraints of the Uzawa theorem, the usefulness of some research disappears if the production schedule becomes too complicated for the researcher to be able to trace deviations in the projected paths of their sources. Suppose, for example, that the researcher were interested in explaining the effect of substitution between energy sources as well as between aggregate energy and other inputs. It would be far easier to point to single valued, exogenous elasticities rather than trends in endogenous elasticities that changed with employment ratios. But would holding at least one elasticity equal to unity undermine the results? Unless one had a belief that the relative shares of energy, on the one hand, or the total share of energy, on the other, were constant, the answer to this question must be yes. And so, the quandary: while imposing constant elasticities might force you into an inappropriate model, choosing a variable elasticity representation can destroy the transparency so essential in understanding the workings of long term projections.

The point of this note is to outline an approximation procedure that was designed to circumvent this quandary in the context just described. It is a general procedure that employs the Uzawa theorem, the source of the problem, to develop a solution. At each point in time, the production schedule is approximated by a Cobb–Douglas representation of the form

$$y = g(x_1, x_2, x_3) = x_1^\alpha [\beta_2 x_2^\rho + (1 - \beta_3) x_3^\rho]^{(1-\alpha)/\rho},$$

but the relative share devoted to x_1 – the α parameter – is systematically adjusted over time so that the long-run elasticities between x_1 and both x_2 and x_3 are not unity. Supply conditions are required to compute shares, of course, but the result is an iterative procedure that allows arbitrary, constant elasticities over the long run. Section 2 will present the intuition behind the procedure with the help of a little geometry, and section 3 will develop its nuts and bolts. A final section then cites the information gleaned by using this approach in the energy modeling that inspired it.

2. The mechanics of the approximation procedure

Suppose that the goal is to represent

$$y(t) = f[x_1(t), x_2(t), x_3(t)]$$

over time with

$$\sigma_{23} = (1-\rho)^{-1} \tag{1a}$$

as usually defined and an elasticity between x_1 and an aggregate of x_2 and x_3 satisfying

$$\sigma' = (1-r)^{-1}. \tag{1b}$$

The idea will be to represent $f(x_1, x_2, x_3)$ by

$$y(t) = x_1(t)^{\alpha(t)}\{\beta_2 x_2(t)^{\rho} + (1-\beta_2)x_3(t)^{\rho}\}^{[1-\alpha(t)]/\rho} \tag{2}$$

during any period t while adjusting the share paid to x_1, $\alpha(t)$, so that the elasticity of substitution between $x_1(t)$ and

$$x(t) = x_1(t) + x_2(t)$$

across time period is $(1-r)^{-1}$.

Condition (1a) is, of course, guaranteed by the structural form of eq. (2). Condition (1b) can be satisfied by construction. Looking closely at

$$y(t) = \{\beta_1 x_1(t)^r + (1-\beta_1)x(t)^r\}^{1/r}$$

in particular, it becomes apparent that

$$w(t)x(t)/w_1(t)x_1(t) = k\{w(t)/w_1(t)\}^{r/(r-1)}, \tag{3}$$

where $k \equiv [(1-\beta_1)/\beta_1]^{\sigma'}$. But if $\alpha(t)$ represents the share paid to $x_1(t)$, then the left-hand side of (3) is simply $[1-\alpha(t)]/\alpha(t)$, and

$$\alpha(t) = \left\{k\{w(t)/w_1(t)\}^{r/(r-1)} + 1\right\}^{-1}, \tag{4}$$

The procedure is now defined. Manipulating $\alpha(t)$ over time according to (4) guarantees an implicit elasticity of substitution between $x_1(t)$ and the aggregate employment of $x_2(t)$ and $x_3(t)$ equal to the desired $\sigma' = (1-r)^{-1}$.

It should be emphasized that the entire construction is based upon equilibrium input prices $w_i(t)$. There must, therefore, be a supply structure operating in the background against which the derived demand schedules developed above can react. With such a structure in place, though, the approximation procedure is an iterative procedure that begins in period t with $\alpha(t)$ in (2), generates $w_i(t)$ and $x_i(t)$ and thus $y(t)$, and then finishes by computing a new $\alpha(t+1)$. To begin with, of course, there must be initial conditions for both the share of x_1 [i.e., $\alpha(0)$] and the (x_1, w_1). To comply with eq. (4), moreover, these initial values must satisfy

$$\alpha(0) = \left\{kw(0)^{r/(r-1)} + 1\right\}^{-1},$$

but that is also easily guaranteed by manipulating k through adjustments in β_1.

3. An application

One of today's most intriguing long term scientific questions involves the 'greenhouse' effect of increasing carbon dioxide concentrations in earth's atmosphere. And since the primary source higher concentrations will be industrial emissions, the question quickly boils down into wondering what will happen to the derived demands for carbon and non-carbon based fuels over the next century or so. To attempt to answer that question in a way that would produce not only a believable scenario

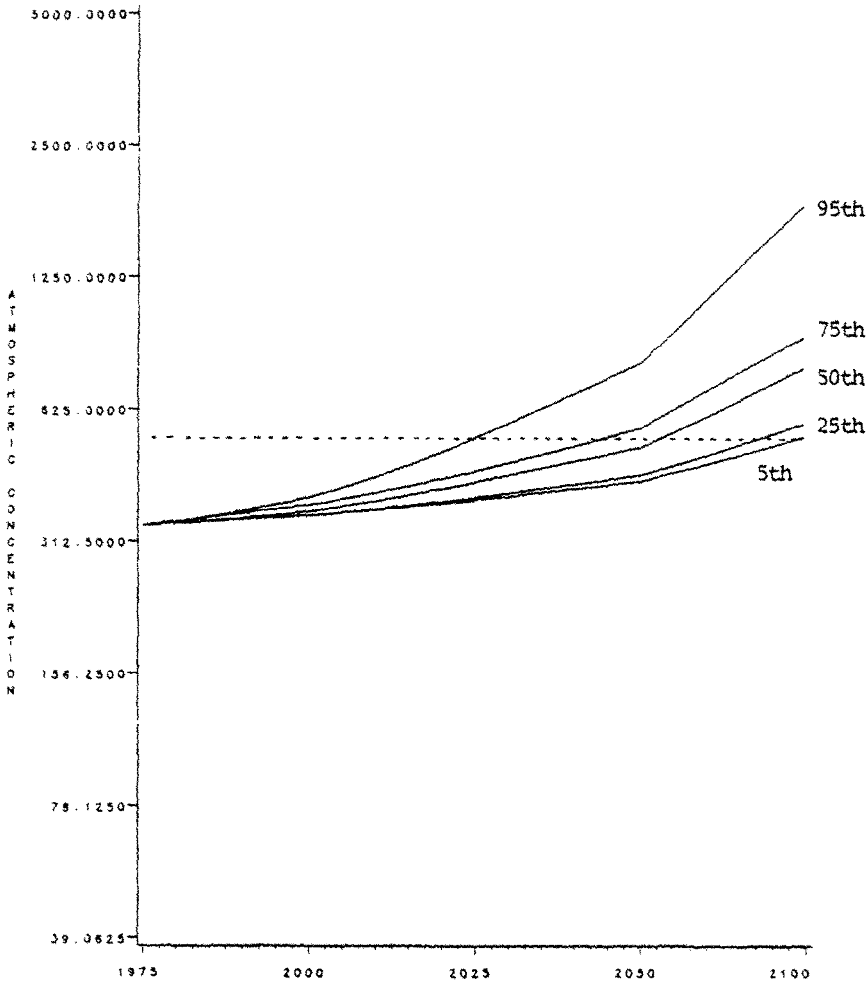

Fig. 1. Plot of atmospheric concentration versus time.

through the year 2100, but also some measure of where the uncertainty surrounding that scenario is generated, Bill Nordhaus and I have constructed a world energy model that incorporates eleven sources of uncertainty. These sources included obvious parameters like population growth estimates, productivity growth estimates and world carbon based energy resources. But because the model was based on energy demand derived from a world production function in labor, carbon based fuel, and non-carbon based fuel, uncertainty was also reflected in the selection of elasticities of substitution between energy types as well as between energy and all other inputs captured by labor. In as much as we did not want to artificially specify either of these elasticities to be unity, it is clear that we needed the procedure outlined above.

Without delineating the details of the model, it is difficult to record here the entire scope of our results. Nonetheless, fig. 1 illustrates enough of their content to illustrate the need to manipulate both elasticities without restriction. The paths shown there represent 5th, 25th, 50th, 75th and 95th percentile paths for carbon dioxide concentrations drawn from the distribution of subjective uncertainty around our maximum likelihood run. And since the doubling of concentrations from preindustrial levels (the dotted line in fig. 1) will apparently signal the beginning of troublesome rises in sea levels, we were able to attach some significant probability (29%) to impending trouble by 2050. Had we been forced by the Uzawa theorem to impose an elasticity of unity on substitution between energy sources, however, we would have underestimated the uncertainty in year 2100 concentrations by more than 18%. The percentile plots would have been closer together, in other words, and our estimate of the chance of doubling by the year 2050 would have been 18% too low. Put another way, a target for research with high potential for both reducing our uncertainty about greenhouse effects and improving our chances of avoiding trouble would have been missed.

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